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ERGODIC THEORY

OF

G-SPACES

by

RICHARD KENNETH THOMAS

A thesis submitted to the University of Warwick
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ABSTRACT

The thesis is in the form of three papers.

In Paper I, affine transformations of a locally compact group are considered. In Part I, it is assumed that the group is not compact: it is shown that an affine transformation of an abelian or connected group cannot be ergodic unless the transformation is of one exceptional type. An attempt is made to obtain stronger conditions than non-ergodicity.

Part II deals with compact groups: it is shown that an affine transformation of a compact group is ergodic if and only if it has a dense orbit. For a connected group, alternative conditions are given. In particular, it is shown that an affine transformation of a Lie group cannot be ergodic unless the group is a torus.

Papers II and III are concerned with the entropy theory of a measure-preserving transformation. The entropy of a transformation T (denoted by $h(T)$) was introduced by Kolmogorov in 1958 (and later modified

by Sinai) as a 'non-spectral invariant': two transformations cannot be isomorphic unless they have the same entropy. Zero entropy has a special significance. In general, every transformation has a unique part with zero entropy; if this part is trivial, the transformation is said to have completely positive entropy. It is very useful to know that a transformation has completely positive entropy: such transformations are mixing of all orders and invertible transformations are Kolmogorov automorphisms.

Paper II considers the question of completely positive entropy when the measure space of the transformation T is a G -space for a compact separable group G . T is required to σ -commute with G -action: $T.g = \sigma g.T$ for all g in G , where σ is a group endomorphism of G onto G . $T_{\xi(G)}$ denotes the induced transformation on the space of G -orbits. It is proved that if T is weakly mixing (has a continuous spectrum) and $T_{\xi(G)}$ has completely positive entropy, then T has completely positive entropy. This theorem 'lifts' the property of completely positive entropy from the orbit space to the fundamental space.

(iii)

In Paper III, it is shown that under suitable conditions (which are not in fact very restrictive)

$$h(T) = h(T_{\mathfrak{g}(G)}) + h(\sigma) .$$

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FORM OF THESIS

The thesis is in the form of three papers:

PAPER I

Affine transformations of locally compact groups

PAPER II

Metric properties of transformations of G-spaces

PAPER III

The addition theorem for the entropy of
transformations of G-spaces

PAPER I

AFFINE TRANSFORMATIONS OF LOCALLY

COMPACT GROUPS

INTRODUCTION

This paper examines the dynamical system (G, m, T) , where G is a locally compact topological group, m is Haar measure and T is an affine transformation, i.e. $T = aA$, where a is a fixed element of G and A is a group automorphism of G . The main objective is to determine when T can be ergodic.

In Part I, non-compact groups are considered. A result similar to that given in [1] and [2] for group automorphisms is obtained for affine transformations: it is shown that if G is abelian or connected, then T cannot be ergodic unless it is one very exceptional type of transformation. An attempt is made to obtain stronger properties than non-ergodicity.

For compact abelian groups, necessary and sufficient conditions for T to be ergodic have been given by F. Hahn [3], A. H. M. Hoare and W. Parry [4] and P. Walters [5]; Part II of the present paper gives these conditions for general compact groups..

PART I

NON-COMPACT GROUPS

1. ELEMENTARY RESULTS

1.1. DEFINITIONS. For a space of infinite measure, the definition of ergodicity is as follows:

DEFINITION 1. T is ergodic if for any measurable set $E \subset G$, $TE = E \pmod{0}$ implies that $m(E) = 0$ or that $m(G - E) = 0$.

As T is rarely ergodic, other properties are sought; of particular interest is the notion of recurrence (see [6], pp 10-12):

DEFINITION 2. A point x in a measurable set $E \subset G$ is said to be recurrent (with respect to E and T) if $T^n x$ is in E for at least one positive integer n .

The contrary notion is that of a 'wandering' set:

DEFINITION 3. A measurable set E of positive measure is said to be a wandering set for T if the sets E, TE, T^2E, \dots are pairwise disjoint.

DEFINITIONS 4. If there exists a wandering set for T , then T is said to be dissipative; if there is no wandering set, then T is said to be conservative.

If T is conservative, then almost every point of G is recurrent ([6], p 11). The following ideas are easier to work with:

DEFINITIONS 5. T is said to be compressible if there exists a measurable set $E \subset G$ such that $TE \subset E$ and $m(E - TE) > 0$. In the contrary case, T is said to be incompressible.

Compressibility is equivalent to dissipation (and incompressibility to conservation) ([6], p 11). An arbitrary T has an essentially unique incompressible part Y : there exists an invariant measurable set Y ($TY = Y$) such that T is incompressible on Y and there exists a wandering set F such that $G - Y = \bigcup_{n=-\infty}^{\infty} T^n F \pmod{0}$.

DEFINITION 6. If $m(Y) = 0$, then T is said to be totally dissipative. (compressible).

DEFINITION 7. If T preserves the measure m and G contains a sequence $E_1 \subset E_2 \subset \dots$ of compact T invariant sets such that $\bigcup_n E_n = G$, then T will be said to have Property A.

Each E_n in the last definition is of finite measure and so cannot be 'compressed' (since T preserves measure); thus, Property A implies incompressibility.

In the appendix of [7], Yuzvinskii defines yet another concept:

DEFINITION 8. T is said to be weakly ergodic if for any measurable set $E \subset G$, $TE = E$ implies $m(E) = 0$ or $m(E) = \infty$.

Clearly, T is weakly ergodic if T is totally dissipative and measure preserving; Property A denies weak ergodicity.

1.2. THE EXCEPTIONAL CASE. The following example is considered: G is the additive group of the integers and T is translation by one ($Tx = x + 1$). T is measure preserving, ergodic, weakly ergodic and (totally) dissipative. This example is unique in the sense that any transformation that is both ergodic and dissipative must be isomorphic to it, i.e. must have a single infinite orbit of atoms (any proper subset of a wandering set will generate a non-trivial invariant subset of G). The following example shows that the group in such a system does not have to be the integers.

G is taken to be the semi-direct product of Z_2 (the integers mod 2) and the integers: every element of G can be written in the form (x,y) , where x is 0 or 1 and y is an integer; multiplication is given by

$$(x_1, y_1) \cdot (x_2, y_2) = (x_1 + x_2, y_2 + (-1)^{x_2} y_1),$$

addition in the first factor being mod 2.

e.g.

$$(1,5) \cdot (1,6) = (0, 6 - 5) = (0,1).$$

A is given by

$$A(x,y) = (x, x - y)$$

e.g.

$$A(1,1) = (1, 1 - 1) = (1,0); \quad A(0,1) = (0,-1).$$

a is put equal to $(1,0)$. The orbit of $(0,0)$ is:

$$\dots, (1,2), (0,-1), (1,1), (0,0), (1,0), (0,1)$$

$$(1,-1), (0,2), (1,-2), \dots$$

Clearly, this set contains every element of G, i.e. T has a single infinite orbit.

Apart from this type of transformation, T cannot be ergodic unless it is both weakly ergodic and incompressible; it will frequently be found that these two properties are mutually exclusive.

1.3. SUBGROUPS. Suppose that H is an A invariant ($AH = H$) closed normal subgroup of G . $T_{G/H}$ and $A_{G/H}$ will denote the transformations induced on G/H by T and A ; p_H will denote the natural projection of G onto G/H .

In the work that follows, it will be useful to deduce ' T has Property X ' from ' $T_{G/H}$ has Property X '. p_H^{-1} of a wandering set or an invariant set is wandering or invariant respectively and so X can be total dissipation, dissipation or non-ergodicity. p_H^{-1} of a compact set will be compact only if H is compact and so X can be A only when H is compact.

In some cases, it will be impossible to avoid using non-compact invariant subgroups and so for convenience, the following definition is introduced:
 DEFINITION 9. T will be said to have Property B if there exists an A invariant closed normal subgroup H such that $T_{G/H}$ has Property A .

Clearly, Property B denies ergodicity: there are plenty of non-trivial invariant sets.

1.4. NON-MEASURE-PRESERVING TRANSFORMATIONS.

THEOREM. If T does not preserve the (left) Haar measure m , then T is totally dissipative, weakly ergodic and not ergodic.

PROOF. T is an open map and so $m.T$ is a measure equivalent to m and

$$m.T(gE) = m(aAg.AE) = m(AE) = m(aAE) = m.T(E) ,$$

i.e. $m.T$ is left invariant. The uniqueness of Haar measure implies that $m.T = cm$, where c is some positive constant. As T does not preserve m , $c \neq 1$. It is assumed that $c < 1$ (if not replace T by T^{-1}).

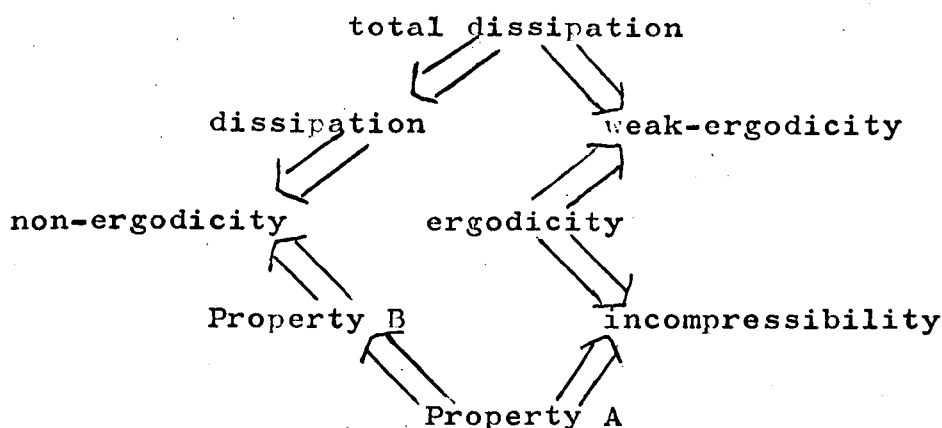
If $TE = E$, then $m(TE) = m(E) = cm(E)$ and so $m(E)$ must be zero or infinity, i.e. T is weakly ergodic.

Let V be any set of finite non-zero measure and let $W = \bigcup_{n=0}^{\infty} T^n V$.

$$m(W) \leq \sum_{n=0}^{\infty} m(T^n V) = \sum c^n m(V) = \frac{1}{1-c} m(V) < \infty .$$

$TW \subset W$ and $m(TW) = cm(W)$ and so $m(W - TW) > 0$, i.e. T is compressible. As V was chosen arbitrarily, T is totally dissipative (compressible). T is clearly not of the exceptional type of §.1.2 and so dissipation implies non-ergodicity.

Using the result of the theorem and ignoring the exceptional case of § 1.2, the relationships between the various properties that have been defined can be displayed as follows:



The strong conditions (apart from ergodicity) are total dissipation and Property A, only one of which can occur in any given system; these will be sought wherever possible.

The theorem above gives a complete solution for an affine transformation that does not preserve left Haar measure. The theorem also applies to right Haar measure. In future, it will always be assumed that T preserves both left and right Haar measures (m can be taken to be either).

1.5. TWO LEMMAS. These will be needed later.

LEMMA 1. If there exists a measurable function f from G to \mathbb{C} such that $m(f^{-1}(0)) = 0$ and

$$f(Tx) = cf(x) ,$$

where c is some constant of modulus not equal to one, then T is totally dissipative.

PROOF. Let E be the measurable set

$\{x \in G: |f(x)| \in (|c|, |c|^2]\}$; E is a wandering set and $\bigcup_{n=-\infty}^{\infty} T^n E = G - f^{-1}(0) = G \bmod 0$. So T is totally

dissipative.

LEMMA 2. If there exist measurable functions f_1 and f_2 from G to \mathbb{C} such that $m(f_1^{-1}(0)) = 0$ and

$$f_1(Tx) = cf_1(x) \quad \dots (1)$$

$$f_2(Tx) = cf_2(x) + f_1(x) , \quad \dots (2)$$

where c is some constant, then T is totally dissipative.

PROOF. For $|c| \neq 1$, lemma is proved by applying

Lemma 1 to f_1 . Assume now that $|c| = 1$.

Let $U = G - f_1^{-1}(0)$. f_1 and f_2 are restricted to U and (2) is divided by (1) to give:

$$\frac{f_2(Tx)}{f_1(Tx)} = \frac{f_2(x)}{f_1(x)} + \frac{1}{c} .$$

After multiplying by c and taking real parts, this becomes:

$$F(Tx) = F(x) + 1 ,$$

where

$$F(x) = \operatorname{Re} \frac{cf_2(x)}{f_1(x)} .$$

Let E be the measurable set $\{x \in G: F(x) \in (0, 1]\}$; E is a wandering set and $\bigcup_{n=-\infty}^{\infty} T^n E = U = G \bmod 0$ and so T is totally dissipative.

1.6. TRANSLATIONS. Let G be any non-compact group and suppose that $Tx = ax$. Let F be the closed subgroup of G generated by a ; F is either compact or discrete. If F is compact, then for any compact neighbourhood U , FU is compact and T invariant and so T has Property A. If F is discrete, then all sufficiently small sets will be wandering and so T is totally dissipative.

1.7. DISCRETE GROUPS. If G is discrete, then every point is either periodic ($T^n x = x$) or is a wandering set. So T either has Property A (all points periodic), or is dissipative (some points wandering) (totally dissipative if all points are wandering but T can easily have a proper incompressible part).

2. ABELIAN GROUPS

2.1. LEMMA. If T is an automorphism of euclidean space R^n , then T is either totally dissipative or has Property A.

PROOF. A coordinate system is chosen in R^n ; a point g is then represented by an n -tuple $\begin{pmatrix} x_1(g) \\ \vdots \\ x_n(g) \end{pmatrix}$ and T

can be considered as a matrix. Let P be the Jordan canonical form of T , i.e. $T = Q^{-1}PQ$, where Q is some non-singular matrix (note that in general, P and Q are complex, whereas T is real) and

$$\begin{pmatrix} y_1(Tg) \\ y_2(Tg) \\ \vdots \\ y_n(Tg) \end{pmatrix} = P \begin{pmatrix} y_1(g) \\ y_2(g) \\ \vdots \\ y_n(g) \end{pmatrix} = \begin{pmatrix} c_1 & & & 0 \\ e_1 & c_2 & & \\ & \ddots & \ddots & \\ 0 & & e_{n-1} & c_n \end{pmatrix} \begin{pmatrix} y_1(g) \\ y_2(g) \\ \vdots \\ y_n(g) \end{pmatrix},$$

where

$$\begin{pmatrix} y_1(g) \\ \vdots \\ y_n(g) \end{pmatrix} = Q \begin{pmatrix} x_1(g) \\ \vdots \\ x_n(g) \end{pmatrix}$$

and $e_i = 0$ or 1 (if $e_1 = 1$, $c_1 = c_2$, etc.).

The y_i 's are linear combinations of the coordinate functions and so it follows that $m(y_i^{-1}(0)) = 0$ for all i . Three cases are considered:

(i) P has an eigenvalue of modulus not equal to one. It can be assumed w.l.g. that $|c_1| \neq 1$.

$y_1(Tg) = c_1 y_1(g)$ and so Lemma 1 of § 1.5 can be applied to give: T is totally dissipative.

(ii) $|c_i| = 1$ for all i and P is not diagonal. It can be assumed w.l.g. that $c_1 = c_2$ and $e_1 = 1$. $y_1(Tg) = c_1 y_1(g)$ and $y_2(Tg) = c_1 y_2(g) + y_1(g)$ and so Lemma 2 of § 1.5 can be applied to give: T is totally dissipative.

(iii) $|c_i| = 1$ for all i and P is diagonal. Let E_m be the measurable set $\{g \in \mathbb{R}^n : |y_i(g)| \leq m \text{ for all } i\}$, $m = 1, 2, \dots$; E_m contains the origin and so is non-empty. E_m is T invariant and as the y_i 's are linear combinations of the coordinate functions, E_m is compact and $m(E_m) > 0$. Also $\bigcup_{m=1}^{\infty} E_m = \mathbb{R}^n$ and so T has Property A.

Thus, T is either totally dissipative or has Property A.

2.2. THEOREM. Let C be the connected component of the identity of the abelian group G . If G/C is compact, then any affine transformation T is either totally dissipative or has Property A.

PROOF. $G = K \oplus \mathbb{R}^n$, where K is some compact group ([8], § 2.4); K is unique and so is A invariant. By § 1.3, it is sufficient to consider $T_{G/K}$, which is an affine transformation on \mathbb{R}^n . It is assumed from now on that G is \mathbb{R}^n .

The map $x \longrightarrow x^{-1}Ax$ is considered: this map is continuous and so the range B is connected; in fact, B is a closed subgroup of \mathbb{R}^n and so $B = \mathbb{R}^m$, $0 \leq m \leq n$, and $G/B = \mathbb{R}^{n-m}$. B is A invariant.

If $a \notin B$, then

$$T(xB) = aAx.B = axx^{-1}Ax.B = ax.B,$$

i.e. $T_{G/B}$ is just translation by $p_B a$. The closed subgroup generated by $p_B a$ is discrete (see § 1.6) and so $T_{G/B}$ and hence T (by § 1.3) is totally dissipative.

If $a \in B$, then $a = b^{-1}Ab$ for some $b \in G$. Thus,

$$Tx = aAx = b^{-1}AbAx = b^{-1}(A(bx)),$$

i.e. T is isomorphic to A ; the theorem follows from

Lemma 2.1.

2.3. DISCONNECTED GROUPS. When G/C is not compact, a result as strong as Theorem 2.2 cannot be expected: a dissipative transformation of a discrete group (§ 1.7) can easily have a non-trivial incompressible part for example.

2.4. LEMMA. If G is a totally disconnected non-compact abelian group, then T is dissipative or has Property A; if T has Property A, then G contains a compact open A invariant subgroup F .

PROOF. Being totally disconnected, G has a compact open subgroup H . Let $M_k = AH.A^2H. \dots .A^kH$ and let $M = \bigcup_{k=1}^{\infty} M_k$. M_k is a compact open subgroup of G and M is an open subgroup of G and therefore closed.

If H is contained in M , then $\{M_k\}$ forms an open cover of H and consequently, $H \subset M_K$ for some K . Then $A(M_K) = A(HM_K) = M_{K+1}$. But $M_K \subset M_{K+1}$ and so M_K must equal M_{K+1} (A is measure preserving), i.e. M_K is an A invariant compact subgroup. By § 1.3, it is sufficient to consider T_{G/M_K} which is an affine transformation of a discrete group: theorem follows from § 1.7. If T has Property A, put $M_K = F$.

Suppose now that H is not contained in M . Let $W = \bigcup_{n=0}^{\infty} T^n M$. $T^n M$ is a coset of $A^n M$ which is an open subgroup of M . Hence, either $T^n M$ is disjoint from M for all $n \geq 1$ or $T^p M$ is contained in M for some p which is chosen as small as possible. $T^p M \subset M$ implies that $T^{np} M \subset T^p M$ for $n = 1, 2, \dots$; otherwise, $T^m M$ is disjoint from M . Thus, the only elements of the set $\{T^m M : m = 1, 2, \dots\}$ contained in M are all in the single coset $T^p M$ of $A^p M$. It follows that the other cosets of $A^p M$ are not in TW and so $m(W - TW) > 0$ implying that T is dissipative.

2.5. THEOREM. If G is a general non-compact abelian group, then T is dissipative or has Property A.

PROOF. If the connected component of the identity C is compact or G/C is compact or $T_{G/C}$ is dissipative, then the theorem follows immediately from § 1.3 and Theorem 2.2. So, using the last lemma, it is assumed that $T_{G/C}$ has Property A. By Lemma 2.4, G/C contains a compact open $A_{G/C}$ -invariant subgroup F/C ; all the orbits of $T_{G/F}$ are finite and so for an element $g \in G$, $T^p(gF) = gF$ for some p which is chosen as small as possible and $T^n(gF)$ is disjoint from gF

for $n = 1, 2, \dots$. T^P is also an affine transformation:

$$T^P = a_p A^P.$$

For an element $x \in F$,

$$T^P(gx) = a_p A^P g A^P x = g(g^{-1} a_p A^P g) A^P x,$$

i.e. T^P on gF is equivalent to the affine transformation $S(g) = (g^{-1} a_p A^P g) A^P$ on F .

By Theorem 2.2, $S(g)$ is totally dissipative or has Property A; T restricted to $\cup T^n(gF)$ has the corresponding property. As $\cup T^n(gF)$ is open in G , T is dissipative if $S(g)$ is totally dissipative for any g and T has Property A if $S(g)$ has Property A for all g .

3. CONNECTED GROUPS

3.1. THEOREM. An affine transformation T of a connected group G is totally dissipative or has Property B.

PROOF. Being connected, G contains a maximal compact normal subgroup H such that G/H is a Lie group with no compact normal subgroups ([9], p 172). H is A invariant and so by § 1.3, it is sufficient to consider $T_{G/H}$. So from now on, G is assumed to be a Lie group with no compact subgroups.

The adjoint representation U is considered: $U(g)$ is the automorphism induced on the Lie algebra of G by the inner automorphism $x \longrightarrow gxg^{-1}$ of the group. $U(g)$ can be considered as a matrix. The kernel of the representation is the centre Z of G ; it will be assumed that G is not abelian (the abelian case was dealt with in § 2.2) in which case G/Z is not compact.

The affect of T on the adjoint representation is as follows:

$$(Tg)x(Tg)^{-1} = aAg.x(Ag)^{-1}a^{-1} = a(A(g(A^{-1}x)g^{-1})a^{-1}.$$

So $U(Tg) = U(a)BU(g)B^{-1}$, where B is the automorphism

of the Lie algebra induced by A; B can be considered as a constant matrix. Let M be the subdiagonal Jordan canonical form of $U(a)B$: $M = PU(a)P^{-1}$; let N be the superdiagonal Jordan form of B: $N = QBQ^{-1}$ and put $V(g) = PU(g)Q^{-1}$. Then

$$PU(Tg)Q^{-1} = MPU(g)Q^{-1}N$$

or

$$V(Tg) = MV(g)N.$$

Written out in full this is:

$$\begin{pmatrix} v_{11}(Tg) & v_{12}(Tg) & \dots \\ v_{21}(Tg) & v_{22}(Tg) & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} = \begin{pmatrix} \lambda_1 & 0 & \dots \\ d_1 & \lambda_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix} \begin{pmatrix} v_{11}(g) & \dots \\ v_{21}(g) & \dots \\ \vdots & \vdots \end{pmatrix} \begin{pmatrix} \mu_1 & e_1 & \dots \\ 0 & \mu_2 & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

where $d_i = 0$ or 1 , $e_j = 0$ or 1 , $\lambda_1 = \lambda_2$ if $d_1 = 1$ etc.

The v_{ij} 's are linear combinations of the elements of $U(g)$ and so either $v_{ij} \equiv 0$ or $m(v_{ij}^{-1}(0)) = 0$. The remainder of the proof consists of picking out functions to which Lemma 1 or Lemma 2 of § 1.5 can be applied. On multiplying out, the right hand side of the above equation becomes:

$$\begin{pmatrix} \lambda_1 \mu_1 v_{11} & \lambda_1 e_1 v_{11} + \lambda_1 \mu_2 v_{12} & \dots \\ d_1 \mu_1 v_{11} + \lambda_2 \mu_1 v_{21} & d_1 e_1 v_{11} + \lambda_2 e_1 v_{21} + d_1 \mu_2 v_{12} + \lambda_2 \mu_2 v_{22} & \dots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

$T_{G/Z}$ (and hence T) will be totally dissipative if any of the following apply:

- a) $v_{11} \neq 0$ and $|\lambda_1 \mu_1| \neq 1$ (apply Lemma 1);
- b) $v_{11} \neq 0$ and $e_1 = 1$ (apply Lemma 2 to $\lambda_1 v_{11}$ and v_{12});
- c) $v_{11} \neq 0$ and $d_1 = 1$ (apply Lemma 2 to $\mu_1 v_{11}$ and v_{21});
- d) $v_{11} = 0$ and $\lambda_2 e_1 v_{21} + d_1 \mu_2 v_{12} \neq 0$ (apply Lemma 2 with $f_1 = \lambda_2 e_1 v_{21} + d_1 \mu_2 v_{12}$ and $f_2 = v_{22}$).

So T is totally dissipative unless $v_{ij}(Tg) = \lambda_i \mu_j v_{ij}(g)$,

$|\lambda_i \mu_j| = 1$, $i, j = 1, 2$. Should this last condition

apply, move on to the block $\begin{matrix} v_{12} & v_{13} \\ v_{22} & v_{23} \end{matrix}$; the condition

on v_{ij} ($i, j = 1, 2$) implies that this block behaves in the same sort of way under T as did the previous block and so this block can be analysed in the same way. If total dissipation is still not implied, continue moving along the first two rows and then, if necessary start on rows 2 and 3, etc. If total dissipation is not implied at any point, then

$v_{ij}(Tg) = \lambda_i \mu_j v_{ij}(g)$, $|\lambda_i \mu_j| = 1$, for all i and j .

If this is the case, invariant compact neighbourhoods in G/Z can be obtained by restricting the moduli of the v_{ij} 's, i.e. $T_{G/Z}$ has Property A and so T has Property B.

5.2. REMARK. It is very probable that an affine transformation of a connected group is totally dissipative or has Property A. The proof of Theorem 3.1 gives this if Z is trivial or compact; for an abelian group this result is given by Theorem 2.2. It can be proved in a number of other situations without too much difficulty but the general case appears to be evasive. The total dissipation/Property A relation is thus a property of connected spaces: it would be interesting to know exactly what conditions are needed for a transformation to possess it.

3.3. GENERAL NON-ABELIAN GROUPS. Results for an affine transformation of a general non-compact group depend on results for a totally disconnected, non-abelian, non-discrete, non-compact group: at present, not enough is known about the structure of such groups to be able to solve the problem - does an automorphism of such a group always have an invariant compact open subgroup?

PART II

COMPACT GROUPS

4. THE SPECTRUM OF T

4.1. REPRESENTATIONS. Representation theory of a compact group will be used (as in [10], Chapter 5 for example). The following clarifies notation:

By a 'representation of G ' will always be meant a continuous unitary representation of G . Every compact group G has a set of irreducible representations, $\{U^\alpha\}$, each of which is finite dimensional and so can be considered as a matrix group: $U^\alpha(g) = (u_{ij}^\alpha(g))$. U^α is said to be equivalent to U^β if $U^\alpha(g) = QU^\beta(g)Q^{-1}$ for all g in G , where Q is some constant matrix.

$[U^\alpha]$ denotes the class of all representations equivalent to U ; $\text{Ker } U^\alpha$ denotes the kernel of U^α : $\text{Ker } U^\alpha$ depends only on the class of U^α : $\text{Ker } U^\alpha = \text{Ker } [U^\alpha]$. The continuous functions $\{u_{ij}^\alpha\}_{ij\alpha}$ constitute an orthogonal basis for $L^2(G)$. $F(U^\alpha)$ will denote the

subspace of $L^2(G)$ generated by the elements of U^α .

$F(U^\alpha)$ depends only on the class of U^α : $F(U^\alpha) = F([U^\alpha])$.

The affine transformation $T (= aA)$ acts as a unitary operator on $L^2(G)$: $f(g) \xrightarrow{T} f(Tg) = fT(g)$ - the same symbol is used for the operator as for the transformation but it is written on the right.

4.2. TYPES OF SPECTRUM.

DEFINITION. If H_0 is a T invariant subspace of $L^2(G)$ ($H_0 T = H_0$), then T_{H_0} (T restricted to H_0) is said to have discrete spectrum if there exists a complete orthogonal basis $\{e_i\}$ for H_0 such that $e_i(Tg) = c_i e_i(g)$, where c_i is some constant of modulus one.

DEFINITION. If H_1 is a T invariant subspace of $L^2(G)$, then T_{H_1} is said to have Lebesgue spectrum if there exists a complete orthogonal basis $\{f_{i,j}\}$ for H_1 such that $f_{i,j} T = f_{i,j+1}$.

4.3. THEOREM. $L^2(G) = H_0 + H_1$, where T_{H_0} has discrete spectrum and T_{H_1} has Lebesgue spectrum.

The basis functions for H_0 and H_1 can all be taken to be continuous functions on G .

PROOF. For an irreducible representation U^α , $U^\alpha A^n$ is also an irreducible representation and there are two possibilities:

(i) $U^\alpha A^{r(\alpha)}$ is in $[U^\alpha]$ for some positive integer $r(\alpha)$ which is chosen as small as possible;

(ii) $U^\alpha A^n$ is not in $[U^\alpha]$ for any n in which case $r(\alpha)$ is put equal to ∞ .

T^n is affine: $T^n = a_n A^n$ and so $F(U^\alpha T^n) = F(U^\alpha A^n)$. Let $E(U^\alpha) = \bigcup_{n=-\infty}^{\infty} F(U^\alpha T^n)$: $E(U^\alpha)$ is T invariant.

If $r(\alpha) < \infty$, then $E(U^\alpha)$ is finite dimensional and as T is unitary, $T_{E(U^\alpha)}$ has a complete set of orthogonal eigenfunctions $\{e_i^\alpha\}$: $e_i^\alpha(Tg) = c_i^\alpha e_i^\alpha(g)$, $|c_i^\alpha| = 1$. The e_i^α 's are linear combinations of $u_{ij}^\alpha T^n$'s and so are continuous functions on G .

H_0 is put equal to $\bigcup_{\alpha} \{E(U^\alpha) : r(\alpha) < \infty\}$.

If $r(\alpha) = \infty$, then the set $\{u_{ij}^\alpha T^n : n = 0, \pm 1, \pm 2, \dots\}$ is pairwise orthogonal and so $T_{E(U^\alpha)}$ has Lebesgue spectrum. H_1 is put equal to $\bigcup_{\alpha} \{E(U^\alpha) : r(\alpha) = \infty\}$.

H_0 always contains the one-dimensional subspace of constant functions $F(U^0) = E(U^0)$, U^0 denotes the constant representation throughout - $c^0 = 1$.

H_1 can be empty.

4.4. ERGODICITY CONDITIONS. T is ergodic if and only if the only T invariant L^2 function on G is constant a.e. Clearly, a T invariant function must be orthogonal to H_1 (as in the theorem) and so is a linear combination of eigenfunctions; such a function is T invariant if and only if the corresponding eigenvalues are all equal to one. So T is ergodic if and only if $c_i^\alpha = 1$ implies that $\alpha = 0$.

4.5. THEOREM. If G has a countable topological base, then T is ergodic if and only if T has a dense orbit, i.e. for some point g_0 in G , the topological closure of the set $\{T^n g_0\}$ is equal to G .

PROOF. Suppose that $\overline{\{T^n g_0\}} = G$ for some g_0 and that $c_i = 1$. Then $e_i^\alpha(T^n g_0) = e_i^\alpha(g_0)$ for all n . As e_i is continuous on G , e_i^α must be constant, i.e. $\alpha = 0$ and so T is ergodic.

The converse is a special case of a well known result ([6], Lemma p 26): almost every orbit of an ergodic measure-preserving transformation of a topological space with a countable base is dense.

For G abelian, this result was proved by P. Walters ([5], Theorem 1).

5. CONNECTED GROUPS

Throughout this section, it will be assumed that G has a countable topological base.

5.1. THEOREM. If G is a connected Lie group, then T can be ergodic only if G is a torus.

PROOF. G is of the form $\frac{C \otimes D}{N}$, where C is a connected abelian group isomorphic to some subgroup of the centre Z of G , D is a finite direct product of simple Lie groups and N is a subgroup of the centre of $C \otimes D$ ([11]). Clearly, $Z = \frac{C \otimes Z'}{N}$, where Z' is the centre of D . So G/Z is isomorphic to D/Z' which is a semi-simple Lie group. In particular, the fundamental group of G/Z is finite. ([10]). $A_{G/Z}$ is an automorphism of a semi-simple Lie group and so is an isometry (with respect to the left and right invariant metric in G/Z : follows from the fact that the induced automorphism on the Lie algebra of G is an isometry, [10]) and hence, $T_{G/Z}$ is an isometry.

Suppose that T is ergodic. Then $T_{G/Z}$ is also ergodic and so has a dense orbit $\{T_{G/Z}^n s_0\}$. By a well known theorem of Halmos and von Neumann, G/Z can be

made into a torus by defining a (new) multiplication on G/Z by putting $T_{G/Z}^m g_0 \cdot T_{G/Z}^n g_0 = T_{G/Z}^{m+n} g_0$ and extending continuously to the whole of G/Z . So G/Z is topologically a torus (of dimension q say) with a fundamental group \mathbb{Z}^q . This leads to a contradiction unless $q = 0$, in which case, G/Z is a point and so the proof is complete.

5.2. LEMMA (Hoare and Parry [4], Theorem 3). If G is connected and abelian and T is ergodic, then $r(\alpha) < \infty$ implies that $U^\alpha A = U^\alpha$ ($r(\alpha) = 1$).

PROOF. In this case, the U^α 's are one-dimensional (characters) and U^α equivalent to U^β implies that $U^\alpha = U^\beta$. As G is connected, all the characters are of infinite order ($(U^\alpha)^n \neq U^0$ for any n for all $\alpha \neq 0$). Suppose that T is ergodic and that U^α is a character for which $r(\alpha) < \infty$. Consider the function

$$f(g) = \overline{U^\alpha(g)} \cdot U^\alpha A(g) + \overline{U^\alpha(Tg)} \cdot U^\alpha A(Tg) + \dots + \\ + \overline{U^\alpha(T^{r(\alpha)-1}g)} \cdot U^\alpha A(T^{r(\alpha)-1}g)$$

($\overline{}$ denotes complex conjugation). f is a linear combination of distinct non-trivial characters unless $r(\alpha) = 1$. f is T invariant and so must be constant a.e.

implying that a linear combination of distinct characters is constant. This is not possible and so $r(\alpha)$ must be equal to one.

5.3. THEOREM. If G is connected, then T is ergodic if and only if

(i) $r(\alpha) < \infty$ implies that U is one-dimensional and $U^\alpha A = U^\alpha$ ($r(\alpha) = 1$);

(ii) $\{p_B(a^n)\}$ is dense in G/B , where B is the closed normal subgroup of G generated by the elements $\{g^{-1}Ag: g \in G\}$ (p_B denotes the natural projection of G onto G/B).

PROOF. B is A invariant and so

$$T(gB) = aAgB = ag(g^{-1}Ag)B = agB,$$

i.e. $T_{G/B}$ is just multiplication by $p_B a$. Note that condition (ii) implies that G/B is abelian.

Suppose that T is ergodic. Then $T_{G/B}$ is ergodic and by Theorem 4.5, has a dense orbit $\{T_{G/B}^n g_0\} = \{(p_B a)^n g_0\}$. Multiplying on the right by g_0^{-1} this becomes: $\{p_B(a^n)\}$ is dense in G/B .

Suppose that U^α is a representation for which $r(\alpha) < \infty$. Let $M = \bigcap_{n=1}^{r(\alpha)} \text{Ker } U^\alpha A^n$. M is an A invariant closed normal subgroup of G and G/M is a Lie group.

So $T_{G/M}$ is an ergodic affine transformation of a connected Lie group. By Theorem 5.1, G/M must be abelian. Hence, U is one-dimensional and $U^\alpha A = U^\alpha$ by Lemma 5.2.

Now, conversely, suppose that (i) and (ii) are satisfied. By § 4.4, it is only necessary to consider the U^α 's for which $r(\alpha) < \infty$ (in this case $r(\alpha) = 1$); these form a basis for H_0 (as in Theorem 4.3). For such a U^α , $U^\alpha A = U^\alpha$ implies that $U^\alpha(g^{-1}Ag) = 1$: it follows that $U^\alpha(B) = 1$. Thus U^α is essentially a character on G/B . $T_{G/B}$ has a dense orbit and so is ergodic by Theorem 4.5. Consequently, H_0 contains no T invariant ($T_{G/B}$ invariant) functions other than constants and so T is ergodic.

For G abelian, this result was proved by A. H. M. Hoare and W. Parry ([4], Theorem 4).

6. DISCONNECTED GROUPS

6.1. FINITE GROUPS. When G is not connected, the conditions for the ergodicity of T are rather less restrictive than Theorem 5.3: consider the following: G is S_3 which has elements I, P, P^2, Q, PQ, QP ; $P^3 = Q^2 = I, P^2Q = QP$; T is multiplication on the left by P and on the right by Q . T is ergodic: T has orbit I, PQ, P^2, Q, P, QP . All representations of G must be periodic under A ($AX = QXQ$) but are not all one dimensional since G is not abelian and so condition (i) of Theorem 5.3 is not satisfied. Any affine transformation of a finite group is ergodic only if it is of this type, i.e. has a single orbit.

6.2. AN EXAMPLE. The example given now shows that even when G/C is finite (C being the connected component of the identity) the group structure of G does not have to be trivial ($G/C \otimes C$ for example) when T is ergodic - the transformation is of a simple nature of course. G/C will be S_3 as in § 6.1; this group can be represented as a matrix group:

$$I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad P = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \quad Q = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \quad PQ = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \text{ etc.}$$

C is given by matrices of the form $\begin{pmatrix} x & 0 & 0 \\ 0 & y & 0 \\ 0 & 0 & z \end{pmatrix}$, where

x, y and z are complex numbers of modulus one: C is just the three torus. The form of G is now clear: the elements of C are multiplied by the matrices above.

T is given by :

$$TX = \begin{pmatrix} 0 & 0 & a \\ b & 0 & 0 \\ 0 & c & 0 \end{pmatrix} X \begin{pmatrix} 0 & 0 & p \\ 0 & q & 0 \\ r & 0 & 0 \end{pmatrix}, \text{ where } a, b, c, p, q, r$$

are integrally independent ($a^n b^m = 1$ implies $n = m = 0$, etc.) complex numbers of unit modulus. It is easy to see that T^6 is an ergodic rotation on the torus C and so it follows that T is ergodic.

6.3. ERGODICITY CONDITIONS. For G not connected, ergodicity conditions of the kind given in Theorem 5.3 are very unpleasant: this paper will be content to leave them as in § 4.4 and § 4.5.

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PAPER II

METRIC PROPERTIES OF TRANSFORMATIONS

OF G-SPACES

INTRODUCTION

It is very useful to know that a measure preserving transformation T has completely positive entropy; if this is the case, then T is mixing of all orders; if, in addition, T is invertible, then T is a Kolmogorov automorphism. An account of all this can be found in Rohlin's survey article [3]. The present paper considers completely positive entropy when the basic measure space (M, \mathcal{B}, μ) is also a G -space for a compact separable group G . To be precise, the following theorem is proved:

THEOREM A. Let T be a measure preserving transformation of a Lebesgue space (M, \mathcal{B}, μ) which is also a G -space for a compact separable group G . If T satisfies the following conditions:

(i) T is weakly mixing

(has continuous spectrum, see [4], p 39);

(ii) T σ -commutes with G -action, i.e. $Tg = \sigma gT$ for all g in G , where σ is a group endomorphism of G onto G (see § 2);

(iii) $T_{\zeta(G)}$ has completely positive entropy

(see § 1),

then T has completely positive entropy.

This theorem 'lifts' the property of having completely positive entropy from the factor-transformation $T_{\zeta(G)}$ on the space of G -orbits to the transformation T itself. The concepts and notation used in stating the theorem will be considered in more detail in Sections 1 and 2; Section 3 proves the theorem for certain types of group endomorphism σ ; Section 4 completes the proof and Section 5 considers some applications.

An important corollary to Theorem A is that an ergodic group endomorphism has completely positive entropy (see § 5.1). This result was finally proved by Yuzvinskii [1] in 1965, the abelian case having been proved by Rohlin [2] the previous year. Some of Rohlin's and Yuzvinskii's results on the structure of compact groups are employed (for convenience proofs of these are given in an appendix) but the notion of a G -space leads to a simpler proof and a

more general result: Theorem A can be applied to automorphisms of nilmanifolds for example (see § 5).

1. ENTROPY THEORY

It will be assumed that the reader is acquainted with the entropy theory of measure preserving transformations. The purpose of the brief summary given below is to clarify notation and to state results needed later; all references are to Rochlin's survey article [3].

1.1. LEBESGUE SPACES AND PARTITIONS. The fundamental measure space $(M; \mathcal{B}, \mu)$ is a Lebesgue space (normalized so that $\mu(M) = 1$), i.e. the non-atomic part of (M, \mathcal{B}, μ) is measure theoretically point isomorphic to the unit interval with Lebesgue sets and measure. All partitions of M will be assumed to be measurable. A partition ξ generates a sub-algebra $\hat{\xi}$ of \mathcal{B} and conversely, an arbitrary sub-algebra $\hat{\xi}$ is generated by a unique partition ξ (uniqueness, equality, etc. are all considered mod 0). The space M/ξ consisting of elements of ξ together with $\hat{\xi}$ and μ restricted to $\hat{\xi}$ is also a Lebesgue space.

The set of partitions is partially ordered by putting $\xi \leq \eta$ if $\hat{\xi} \subset \hat{\eta}$; ξ denotes the maximal

partition, i.e. the partition of M into distinct points and \mathcal{V} denotes the minimal partition, i.e. the trivial partition whose only element is M itself.

For a sequence of partitions, ξ_1, ξ_2, \dots ,

$\bigwedge_n \xi_n$ denotes the largest partition ξ such that $\xi \leq \xi_n$ for all n and $\bigvee_n \xi_n$ denotes the smallest partition η such that $\eta \geq \xi_n$ for all n .

1.2. TRANSFORMATIONS. For a measure preserving transformation T and a partition ξ , $T^{-1}\xi$ denotes the partition whose elements are the inverse images of the elements of ξ . ξ is said to be T invariant if $T^{-1}\xi \leq \xi$ and to be completely T invariant if $T^{-1}\xi = \xi$. If ξ is T invariant, then T induces a measure preserving factor-transformation T_ξ on M/ξ . T_ξ is invertible if and only if ξ is completely T invariant.

For any measure preserving transformation T , there exists a unique maximal completely T invariant partition $\alpha(T) = \bigwedge_{n=0}^{\infty} T^{-n}\xi$ ([3], § 3.5). Clearly, T_α is invertible.

1.3. COMPLETELY POSITIVE ENTROPY. The entropy of a measure preserving transformation T is denoted by $h(T)$ ([3], § 9). In the set of partitions $\{\xi: h(T_\xi) = 0\}$ there exists a maximal partition $\pi(T)$ known as Pinsker's partition ([3], § 11.5). $h(T_\eta) = 0$ if and only if $\eta \leq \pi(T)$; $\pi(T)$ is completely T invariant and so $\pi(T) \leq \alpha(T)$; for any integer $n \geq 1$, $\pi(T^n) = \pi(T)$ and if T is invertible, $\pi(T^{-1}) = \pi(T)$.

T is said to have completely positive entropy if $\pi(T) = \mathcal{V}$ or equivalently, if every non-trivial factor-transformation has positive entropy.

If $\xi_1 \leq \xi_2 \leq \dots$ is an increasing sequence of T invariant partitions such that $\bigvee_n \xi_n = \mathcal{E}$ and T_{ξ_n} has completely positive entropy for all n , then T has completely positive entropy ([3], § 13.4).

1.4. BERNOULLI AUTOMORPHISMS. An important example of a class of transformations having completely positive entropy is given now. The measure space M is the direct product of a two-way infinite sequence of copies of a Lebesgue space X known as the state space. A point of M is given by a sequence $\{x_n\}_{-\infty}^{\infty}$ and T is defined by $T\{x_n\} = \{y_n\}$, where $y_n = x_{n+1}$.

This system is known as a Bernoulli automorphism.

Of particular interest in the present paper is the case where X is a compact separable group; M is then a compact separable group also and T is a group automorphism. Such a system is known as a Bernoulli group automorphism. All Bernoulli automorphisms have completely positive entropy ([3], § 13.10).

1.5. AUTOMORPHISMS OF TORI. A necessary and sufficient condition for an endomorphism σ of a compact group to be ergodic is that $U_i \sigma^n$ should not be equivalent to U_i for every irreducible unitary representation U_i of the group apart from the constant representation U_0 ([4], p 54). In particular, an ergodic group endomorphism has a continuous spectrum.

When σ is an automorphism of a finite-dimensional torus, σ can be represented as a matrix A with integer entries and determinant ± 1 . The ergodicity condition becomes: A is ergodic if and only if A has no roots of unity as eigenvalues. Sinai [5] and Arov [6] proved that

$$h(A) = \log |\lambda_1| + \dots + \log |\lambda_n| ,$$

where $\lambda_1, \dots, \lambda_n$ are the eigenvalues of A of

modulus greater than one. So $h(A)$ is zero if and only if all the eigenvalues of A lie on the unit circle.

If this is the case, it can be proved that all the eigenvalues are roots of unity. Hence an ergodic group automorphism of a torus has positive entropy ([2], § 3.3).

2. PROPERTIES OF G-SPACES

2.1. DEFINITION. (M, \mathcal{B}, μ) is a G-space, for a compact separable group G, if G acts as a group of invertible measure preserving transformations of M satisfying the following conditions:

$$(i) \quad g.(h.x) = gh.x \quad \text{a.e. for all } g, h \text{ in } G$$

and x in M;

$$(ii) \quad e.x = x \quad \text{a.e., where } e \text{ is the identity of } G;$$

(iii) the representation $g \rightarrow U_g$ induced on $L^2(M)$ by $f(x) \rightarrow f(g^{-1}.x) = U_g f(x)$ is continuous, i.e. for f, h in $L^2(M)$, $\int U_g f. \bar{h} \, d\mu$ is a continuous function from G to \mathbb{C} .

As g is measure preserving, U_g is unitary and so $g \rightarrow U_g$ is a continuous unitary representation of G.

2.2. ORBIT PARTITIONS. Suppose that (M, \mathcal{B}, μ) is a G-space as above and that H is a subgroup of G.

The H-orbit partition $\mathcal{S}(H)$ of M is defined to be the partition generating the algebra

$$\widehat{\mathcal{S}}(H) = \{E \in \mathcal{B} : \mu(gE \Delta E) = 0 \text{ for all } g \text{ in } H\}$$

(Δ denotes symmetric difference).

Orbit partitions have the following, easily verified, properties:

(i) the continuity of the induced representation of G implies that $\xi(H) = \xi(\bar{H})$, where \bar{H} is the topological closure of H ;

(ii) $\xi(e) = \varepsilon$;

(iii) $H_1 \supset H_2$ implies that $\xi(H_1) \leq \xi(H_2)$;

(iv) if $H_1 \supset H_2 \supset \dots$ is a sequence of closed subgroups of G such that $\bigcap_n H_n = e$, then the continuity of the induced representation of G implies that $\bigvee_n \xi(H_n) = \varepsilon$;

(v) similarly, if $H_1 \subset H_2 \subset \dots$ is a sequence of subgroups such that $\bigcup_n H_n = G$, then $\bigwedge_n \xi(H_n) = \xi(G)$.

2.3. σ -COMMUTING PROPERTY. The measure preserving transformation T is said to σ -commute with G -action if $T(g.x) = \sigma(g).Tx$, where σ is a group endomorphism of G . Clearly, $\xi(G)$ is T invariant if this happens.

A subgroup H of G is said to be σ invariant if $\sigma H \subset H$ and to be completely σ invariant if $\sigma H = H$.

If H is σ invariant, then $\xi(H)$ is T invariant.

If H is a σ invariant closed normal subgroup of G , then $M/\xi(H)$ is also a G -space: in fact it is a G/H -space as H acts trivially on $M/\xi(H)$. For such an H , $T_{\xi(H)} \sigma_{G/H}$ commutes with G/H -action on $M/\xi(H)$, where $\sigma_{G/H}$ denotes the endomorphism induced in G/H by σ .

If H is a completely σ invariant closed subgroup of G , then $T \sigma_H$ -commutes with H -action, where σ_H denotes the restriction of σ to H .

If T is invertible, then $T_{\xi(G)}$ is invertible also.

2.4. INVARIANCE OF $\pi(T)$ UNDER G -ACTION. A partition ξ is said to be G invariant if $g\xi = \xi$ for all g in G .

If $T \sigma$ -commutes with G -action and $\sigma(g) = g$ for some g in G , then $g\pi(T)$ (and $g^{-1}\pi(T)$) is completely T -invariant and $T_{g\pi} (T_{g^{-1}\pi})$ is isomorphic to T_π and so has zero entropy implying that $g\pi \leq \pi$ ($g^{-1}\pi \leq \pi$) or equivalently, $\pi \leq g^{-1}\pi$. Hence $\pi(T)$ must be equal to $g\pi(T)$.

Suppose now that σ is densely periodic, i.e. there exists a dense subset K of G such that σ is periodic on every element of K . For any g in K ,

$\sigma^n g = g$ for some n so that

$$g\pi(T) = g\pi(T^n) = \pi(T^n) = \pi(T).$$

Thus $\pi(T)$ is invariant under all the elements of K . The continuity of the induced representation of G implies that $\pi(T)$ is invariant under the whole of G .

2.5. ERGODIC G -ACTION. G -action on (M, \mathfrak{B}, μ) is said to be ergodic if $\xi(G) = \mathcal{V}$, i.e. if $\mu(E \Delta gE) = 0$

for some set E in \mathfrak{B} for all g in G implies that $\mu(E) = 0$ or 1 or equivalently, $U_g f = f$ for some f in $L^2(M)$ and all g in G implies that f is constant a.e.

If a partition ξ of M is G invariant, then M/ξ is also a G -space.

LEMMA. If $T_{\xi(G)}$ has completely positive entropy and $\pi(T)$ is G invariant, then G acts ergodically on $M/\pi(T)$.

PROOF. Let $\xi(G, \pi)$ be the G -orbit partition of $M/\pi(T)$. By § 1.5, $h(T_{\xi(G, \pi)}) = 0$ (since $\xi(G, \pi) \leq \pi(T)$).

But $\hat{\xi}(G, \pi) \subset \hat{\xi}(G)$, showing that $T_{\xi(G, \pi)}$ is a factor-transformation of $T_{\xi(G)}$ and so has completely positive entropy. Hence $\xi(G, \pi) = \mathcal{V}$, i.e. G acts ergodically on $M/\pi(T)$.

ASSUMING THAT $\pi(T) = \mathbb{Z}$. Suppose now that T satisfies the conditions of Theorem A and that $\pi(T)$ is G invariant. The weakly mixing property of T carries over to T_π ; T_π σ -commutes with G -action on $M/\pi(T)$; by the last lemma, $T_{\mathbb{Z}(G, \pi)}$ is trivial (and so has completely positive entropy), i.e. T_π satisfies the requirements of Theorem A. It is required to prove that $\pi(T) = \mathbb{V}$ or equivalently, $M/\pi(T)$ is trivial (consists of a single atom).

So once it has been established that $\pi(T)$ is G invariant, only $M/\pi(T)$ has to be considered. By replacing M by $M/\pi(T)$, T by T_π , it can be assumed that $\pi(T) = \mathbb{Z}$. The proof of Theorem A is then reduced to showing that M is trivial.

2.6. ADJUSTING σ IN THEOREM A. At certain points in the proof of Theorem A, it will be convenient to be able to replace the endomorphism σ by

- (i) an automorphism;
- (ii) σ^n , n being an integer > 1 ;
- (iii) σ^{-1} (σ here must be an automorphism).

This is done as follows:

(i) It is assumed that T satisfies the conditions of Theorem A. Let P be any set in $\widehat{T^{-1}\mathcal{E}}$; $P = T^{-1}Q$, where Q is some set in \mathcal{B} . Clearly, $gP \subset T^{-1}(\sigma g.Q)$. On the other hand, let x be any point in $T^{-1}(\sigma g.Q)$; $Tx = \sigma g.y$, y some point in Q . $T(g^{-1}x) = y \in Q$, i.e. $g^{-1}x \in T^{-1}Q = P$. So x is in gP . Therefore $gP = T^{-1}(\sigma g.Q)$ and so $gP \in \widehat{T^{-1}\mathcal{E}}$; it follows that $T^{-1}\mathcal{E}$ is G invariant. For h in the closed normal subgroup $\sigma^{-1}e$, $hP = P$ and so it follows that $\sigma^{-1}e$ acts trivially (leaves almost all points fixed) on $M/T^{-1}\mathcal{E}$. Similarly, T^{-n} is G invariant and $\sigma^{-n}e$ acts trivially on $M/T^{-n}\mathcal{E}$. Consequently, $\alpha(T) = \bigwedge_{n=0}^{\infty} T^{-n}\mathcal{E}$ is G invariant and the closed normal subgroup $F = \bigcup_{n=1}^{\infty} \sigma^{-n}e$ acts trivially on $M/\alpha(T)$.

As $\pi(T) \leq \alpha(T)$, T has completely positive entropy if and only if T_{α} has and so it is sufficient to consider the invertible transformation T_{α} , $\sigma_{G/F}$ -commuting with G/F -action on $M/\alpha(T)$. $\sigma_{G/F}$ is a group automorphism.

(ii) If T satisfies the conditions of Theorem A, then T^n will satisfy these conditions also but with σ^n in place of σ . As $\pi(T^n) = \pi(T)$, T and σ can be replaced by T^n and σ^n .

(iii) If T is invertible and satisfies the conditions of Theorem A, σ being an automorphism, then T^{-1} will satisfy these conditions also but with σ^{-1} in place of σ . As $\pi(T^{-1}) = \pi(T)$, T and σ can be replaced by T^{-1} and σ^{-1} .

3. PROOF OF THEOREM A FOR CERTAIN SPECIAL CASES

In this section, Theorem A is proved for four types of group automorphism σ ; T is assumed to be invertible (§ 2.6, (i)).

3.1. THEOREM. If T satisfies the conditions of Theorem A with G acting ergodically on M , then either

(i) M is trivial or

(ii) T has Lebesgue spectrum in the orthogonal complement of the invariant subspace of constant functions.

PROOF. Representation theory is used; the terminology of [9], Chapter I is employed: the compactness of G implies that the representation U induced on $L^2(M)$ by G -action is discretely decomposable into a direct sum of primary representations, i.e. $L^2(M) = \sum H_i$, where U restricted to H_i can be represented as $\lambda_i U_i$, where U_i is an irreducible unitary representation which is unique to within equivalence and λ_i is its multiplicity (possibly infinite at this stage). This decomposition is unique and H_i is orthogonal to H_j for $i \neq j$ (U_i is not equivalent to U_j for $i \neq j$). It will be convenient to write $H(U_i)$ for H_i .

An orthonormal set of basis vectors

$e_{i11}, e_{i12}, \dots, e_{i1n}, e_{i21}, \dots, e_{i\lambda_i 1}, \dots, e_{i\lambda_i n}$
(n being the dimension of the representation) is
chosen in Π_i such that:

$$\begin{pmatrix} e_{ij1}(g^{-1}x) \\ \vdots \\ e_{ijn}(g^{-1}x) \end{pmatrix} = U_i(g) \begin{pmatrix} e_{ij1}(x) \\ \vdots \\ e_{ijn}(x) \end{pmatrix}$$

for $j = 1, 2, \dots, \lambda_i$. The column of basis functions
can be considered as a vector valued function

$E_{ij}: M \rightarrow \mathbb{C}^n$ satisfying the equation

$$E_{ij}(g^{-1}x) = U_i(g)E_{ij}(x).$$

The usual \mathbb{C}^n scalar product is considered:

$$\begin{aligned} (E_{ij}(g^{-1}x), E_{ik}(g^{-1}x)) &= (U_i(g)E_{ij}(x), U_i(g)E_{ik}(x)) = \\ &= (E_{ij}(x), E_{ik}(x)), \end{aligned}$$

i.e. the scalar product is a measurable function
which is invariant under G -action. The ergodicity of
the latter implies that the scalar product is
constant a.e.

But

$$\begin{aligned} \int_M (E_{ij}, E_{ik}) d\mu &= \int_M (e_{ij1} \bar{e}_{ik1} + \dots + e_{ijn} \bar{e}_{ikn}) d\mu \\ &= \begin{cases} 0 & \text{if } j \neq k \\ n & \text{if } j = k \end{cases}, \end{aligned}$$

Hence the E_{ij} 's are orthogonal a.e. As there cannot be more than n orthogonal vectors at a point, $\lambda_i \leq n$.

Under G -action, the composition of E_{ij} with T behaves as follows:

$$E_{ij}T(g^{-1}x) = E_{ij}(\sigma(g^{-1})Tx) = U_i(\sigma g)E_{ij}T(x).$$

So the functions $e_{i11}^T, \dots, e_{i i n}^T$ are in $H(U_i \sigma)$.

If $U_i \sigma^m$ were equivalent to U_i for some m and some $i \neq 0$ (U_0 denotes the constant representation throughout), then $H(U_i) + H(U_i \sigma) + \dots + H(U_i \sigma^{m-1})$ would be a finite dimensional subspace of $L^2(M)$ which remains invariant under T ; T would then have a non-constant eigenfunction contradicting the assumption of continuous spectrum.

Thus either

(i) $U = U_0$ implying that M is trivial or

(ii) $U_i \sigma^m$ is not equivalent to U_i ($i \neq 0$) for any m , in which case, basis elements $\{f_{i,j}\}$ can be chosen in $L^2(M)$ such that $f_{i,j} T = f_{i,j+1}$, i.e. T has Lebesgue spectrum in the complement of $H(U_0)$.

3.2. PROOF OF THEOREM A FOR A FINITE GROUP G .

Any automorphism σ is periodic in this case. So $\pi(T)$ is G invariant (§ 2.4) and it can be assumed that $\pi(T) = \mathbb{C}$ (G acts ergodically; § 2.5). Theorem 3.1 is applied: $U_i \sigma^m$ must be equivalent to U_i for some m for all i and so M must be trivial, completing the proof.

3.3. PROOF OF THEOREM A FOR σ AN AUTOMORPHISM OF A FINITE DIMENSIONAL TORUS G . σ is periodic on the roots of the identity and these are dense in G .

So, as in the previous proof, $\pi(T)$ is G invariant and it can be assumed that $\pi(T) = \mathbb{C}$ (G acts ergodically on M). Theorem 3.1 is applied: if (i) applies, the proof is complete and so it is assumed that (ii) applies, i.e. $U \neq U_0$ and $U_i \sigma^m$ is not equivalent to U_i for any m for all $i \neq 0$. As G is abelian, all the irreducible representations are one-dimensional and so it follows that all the H_i 's in the proof of

Theorem 3.1 are one-dimensional. A vector e_i of unit modulus is chosen from each H_i to form an orthonormal basis for $L^2(N)$. The G invariance of the \mathbb{C}^1 scalar product implies that $|e_i(x)| \equiv 1$. Thus for e_i and e_j , $e_i \cdot \bar{e}_j$ is also in $L^2(N)$ and

$$e_i \bar{e}_j(g^{-1}x) = U_i(g) \cdot \bar{U}_j(g) \cdot e_i \bar{e}_j(x)$$

(regarding the U_i 's simply as complex valued functions on G). So the U_i 's form a group B (under pointwise multiplication) and $e_i \bar{e}_j$ is in $H(U_i U_j)$. B is a subgroup of the dual of G and so is the dual of some torus G/H , where H is a completely σ invariant closed subgroup. B is a free group on a finite number of generators, U_1, U_2, \dots, U_n say. By § 1.5, $\sigma_{G/H}$ can be represented as a matrix $A = \{a_{ij}\}$:

$$U_i A = U_1^{a_{1i}} \cdot U_2^{a_{2i}} \cdot \dots \cdot U_n^{a_{ni}}, \quad i = 1, 2, \dots, n.$$

As $e_i T$ is in $H(U_i A)$,

$$e_i T = \mu_i e_1^{a_{1i}} \cdot e_2^{a_{2i}} \cdot \dots \cdot e_n^{a_{ni}}, \quad i = 1, \dots, n,$$

where the μ_i 's are constants of unit modulus depending only on the choice of the e_i 's. It is shown now that the μ_i 's can be removed. For a set of constants

$\lambda_1, \lambda_2, \dots, \lambda_n$ of unit modulus, f_i is put equal to $\bar{\lambda}_i e_i$ and substituted into the last set of equations:

$$\lambda_i f_i T = \mu_i \lambda_1^{a_{1i}} \cdot \lambda_2^{a_{2i}} \cdot \dots \cdot \lambda_n^{a_{ni}} f_1^{a_{1i}} f_2^{a_{2i}} \cdot \dots \cdot f_n^{a_{ni}}.$$

By § 1.5, A is ergodic and $\det(A - I) \neq 0$ and so the equations $\lambda_i = \mu_i \lambda_1^{a_{1i}} \cdot \lambda_2^{a_{2i}} \cdot \dots \cdot \lambda_n^{a_{ni}}$

have a solution $\lambda_i = \alpha_i$ (take logs). For this choice of the λ_i 's,

$$f_i T = f_1^{a_{1i}} \cdot f_2^{a_{2i}} \cdot \dots \cdot f_n^{a_{ni}}, \quad i = 1, \dots, n.$$

Let F be the group generated by these f_i 's; the elements of F constitute a complete orthonormal basis for $L^2(M)$. The map $w: F \rightarrow B$ taking f_i to U_i is a group isomorphism satisfying $w(f_i T) = (w f_i) A$.

Extending first to finite linear combinations and then passing to the limit, w becomes a multiplicative isometry between $L^2(M)$ and $L^2(G/H)$. The following diagram commutes:

$$\begin{array}{ccc} L^2(M) & \xrightarrow{T} & L^2(M) \\ \downarrow w & & \downarrow w \\ L^2(G/H) & \xrightarrow{A} & L^2(G/H) \end{array}.$$

In other words, T and A are equivalent; as W is multiplicative, T and A are also conjugate ([4], pp 44 and 45). Hence $h(T) = h(A)$ ([3], § 16.3) which is positive by § 1.5. This contradicts the assumption $\pi(T) = \xi$ unless $\xi = \psi$. So M must be trivial.

5.4. PROOF OF THEOREM A FOR σ AN AUTOMORPHISM OF A SIMPLE LIE GROUP G . For a simple Lie group, the subgroup of inner automorphisms is of finite index in the group of all automorphisms ([8], p 193).

So, by replacing T and σ by T^n and σ^n if necessary, it can be assumed that σ is an inner automorphism:

$\sigma(y) = gyg^{-1}$ for some element g in G . Putting

$T' = g^{-1}T$, $T = gT' = T'g$. Let K be the closed abelian subgroup of G generated by g ; K is completely σ invariant. G -action commutes with T' and K -action commutes with T so that $\pi(T')$ is G invariant and $\pi(T)$ is K invariant by § 2.4. Clearly $T\xi(K) = T'\xi(K)$.

It is shown that T' has completely positive entropy, from which it will follow that $T'\xi(K)$ and $T\xi(K)$ have. As in the previous proofs, it is assumed that $\pi(T') = \xi$. Theorem 3.1 is applied: as σ is an

inner automorphism, $U_i \sigma$ is equivalent to U_i for all i and so (i) applies showing that $\pi(T') = \nu$.

M can be considered now as a K -space: $T \sigma_K$ -commutes with K -action on M and $T_{\xi(K)}$ has completely positive entropy. Completion of the proof depends on a proof of Theorem A for the automorphism σ_K of the abelian group K ; such automorphisms will be dealt with separately in the next section.

3.5. PROOF OF THEOREM A FOR σ A BERNOULLI GROUP AUTOMORPHISM OF G . σ is easily seen to be densely periodic and so $\pi(T)$ is G invariant and as before, it is assumed that $\pi(T) = \varepsilon$ and $\xi(G) = \nu$.

For $n = \pm 1, \pm 2, \dots$, let H_n be the closed normal subgroup of G consisting of all sequences $\{x_i\}_{i=-\infty}^{\infty}$ with $x_i = e$ for $i \geq n$. Then, $\sigma H_n = H_{n-1}$;

$$\dots \subset H_{-1} \subset H_0 \subset H_1 \subset \dots; \quad \overline{\bigcup_{n=1}^{\infty} H_n} = G \text{ and } \bigcap_{n=-1}^{-\infty} H_n = e.$$

$T_{\xi(H_0)} \sigma_{G/H_0}$ -commutes with G/H_0 -action on $M/\xi(H_0)$.

The next step is to replace σ_{G/H_0} by an automorphism as in § 2.6, (i):

$$F = \overline{\bigcup_{n=1}^{\infty} \sigma_{G/H_0}^{-n} e} = \overline{\bigcup_{n=1}^{\infty} \sigma^{-n} H_0} = \overline{\bigcup_{n=1}^{\infty} H_n} = G,$$

So $G/F = e$, i.e. G acts trivially on $M/\alpha(T_{\xi(H_0)})$.

As G acts ergodically, $\alpha(T_{\xi(H_0)}) = \mathcal{V}$. Hence $T_{\xi(H_0)}$

has completely positive entropy implying that

$\xi(H_0) = \mathcal{V}$ (since it was assumed that $\pi(T) = \xi$).

Similarly $\xi(H_n) = \mathcal{V}$ for $n = -1, -2, \dots$.

Hence $\xi = \bigvee_{n=-1}^{-\infty} \xi(H_n)$ (§ 2.2, (iv)) is equal to \mathcal{V}

and the proof of Theorem A is complete.

4. COMPLETION OF PROOF

In this section, the proof of Theorem A is extended from the special cases of the last section to the general situation; the tools needed for this extension are given in § 4.1.

4.1. PROOF EXTENSION. It is assumed that T satisfies the conditions of Theorem A.

(i) PROOF BY STEPS. Suppose that G contains a completely σ invariant closed normal subgroup H ; $\xi(H)$ is both T and G invariant. $T_{\xi(H)}$ is weakly mixing and $\sigma_{G/H}$ -commutes with G/H -action on $M/\xi(H)$. So if $\sigma_{G/H}$ is a type of endomorphism for which Theorem A has been proved, then application gives: $T_{\xi(H)}$ has completely positive entropy.

T σ_H -commutes with H -action on M and $T_{\xi(H)}$ has completely positive entropy. Hence, if σ_H is also a type of endomorphism for which Theorem A has been proved, then it will follow that T has completely positive entropy.

Thus Theorem A can be proved for σ by proving it for the two 'steps' $\sigma_{G/H}$ and σ_H .

made into a torus by defining a (new) multiplication on G/Z by putting $T_{G/Z}^m g_0 \cdot T_{G/Z}^n g_0 = T_{G/Z}^{m+n} g_0$ and extending continuously to the whole of G/Z . So G/Z is topologically a torus (of dimension q say) with a fundamental group \mathbb{Z}^q . This leads to a contradiction unless $q = 0$, in which case, G/Z is a point and so the proof is complete.

5.2. LEMMA (Hoare and Parry [4], Theorem 3). If G is connected and abelian and T is ergodic, then $r(\alpha) < \infty$ implies that $U^\alpha A = U^\alpha$ ($r(\alpha) = 1$).

PROOF. In this case, the U^α 's are one-dimensional (characters) and U^α equivalent to U^β implies that $U^\alpha = U^\beta$. As G is connected, all the characters are of infinite order ($(U^\alpha)^n \neq U^0$ for any n for all $\alpha \neq 0$). Suppose that T is ergodic and that U^α is a character for which $r(\alpha) < \infty$. Consider the function

$$f(g) = \overline{U^\alpha(g)} \cdot U^\alpha A(g) + \overline{U^\alpha(Tg)} \cdot U^\alpha A(Tg) + \dots + \\ + \overline{U^\alpha(T^{r(\alpha)-1}g)} \cdot U^\alpha A(T^{r(\alpha)-1}g)$$

($\overline{}$ denotes complex conjugation). f is a linear combination of distinct non-trivial characters unless $r(\alpha) = 1$. f is T invariant and so must be constant a.e.

Z are completely invariant under any endomorphism of G .

4.2. TOTALLY DISCONNECTED GROUPS.

LEMMA (Yuzvinskii [1], § 11.5, see Appendix A).

If ρ is an automorphism of a compact separable totally disconnected group H and A is an open normal subgroup such that $\bigcap_{-\infty}^{\infty} \rho^n A = e$ and H/A is simple, then either

- (i) H is finite or
- (ii) ρ is a Bernoulli group automorphism of H .

THEOREM. If τ is an automorphism of a compact separable totally disconnected group H , then H contains a sequence $H = H_0 \supset H_1 \supset H_2 \supset \dots$ of completely τ invariant closed normal subgroups such that either H_n/H_{n+1} is finite or $\tau_{H_n/H_{n+1}}$ is a Bernoulli group automorphism for all n , and $\bigcap_n H_n = e$.

PROOF. As H is totally disconnected, H contains a sequence $H = E_0 \supset E_1 \supset E_2 \supset \dots$ of open normal subgroups such that $\bigcap_n E_n = e$. H_0/E_1 is finite and so an open normal subgroup F_1 can be found such that $H_0 \supset F_1 \supset E_1$ and H_0/F_1 is simple. Put $H_1 = \bigcap_{-\infty}^{\infty} \tau^n F_1$. H_1 is

completely τ invariant and so, applying the lemma, either H_0/H_1 is finite or τ_{H_0/H_1} is a Bernoulli group automorphism.

$H_1/H_1 \cap E_1$ is finite and so there exists a closed normal subgroup F_2 such that $H_1 \supset F_2 \supset H_1 \cap E_1$ and H_1/F_2 is simple. Put $H_2 = \bigcap_{n=-\infty}^{\infty} \tau^n F_2$ and apply the lemma.

$H_2/H_2 \cap E_1$ is finite, etc. This process is continued to obtain H_3, H_4 etc. After a finite number (r say) of steps, $H_r \subset E_1$: when this happens, replace E_1 by $E_2 - H_r/H_r \cap E_2$ is finite etc., to produce H_{r+1}, H_{r+2}, \dots . In due course E_2 is replaced by E_3 and so on. In this way a sequence H_0, H_1, \dots is produced satisfying the requirements of the theorem.

Theorem A has been proved for an automorphism of a finite group (§ 3.2) and for a Bernoulli group automorphism (§ 3.5) and so the last theorem provides a sufficient sequence of steps to use the proof by steps procedure to prove Theorem A for any automorphism (and hence any endomorphism) of a totally disconnected group.

G/C is totally disconnected and so the step $\sigma_{G/C}$ can be made leaving σ_C . The next step will be $\sigma_{C/Z}$; C/Z is a connected group with a trivial centre (see Appendix B).

4.3. CONNECTED GROUPS WITH TRIVIAL CENTRES. First an automorphism ρ of a semi-simple Lie group with a trivial centre is considered. Some power ρ^n is a direct product of automorphisms of simple Lie groups; the proof by steps procedure with Theorem A for an automorphism of a simple Lie group (§ 3.4) gives Theorem A for ρ^n . Theorem A for ρ follows from § 2.6, (ii). The following lemma provides suitable steps now to give Theorem A for any automorphism (and hence any endomorphism) of a connected group with a trivial centre.

LEMMA (Yuzvinskii [1], § 4.1, see Appendix B).

If τ is an automorphism of a compact separable connected group with a trivial centre, then $\tau = \tau_1 \otimes \tau_2$, where τ_1 is a Bernoulli group automorphism and τ_2 is a direct product of automorphisms of semi-simple Lie groups with trivial centres.

So the $\sigma_{C/Z}$ step can be taken leaving σ_Z .

4.4. ABELIAN GROUPS.

DEFINITION. A (discrete) abelian group Γ is said to be finitely generated with respect to an endomorphism ρ if every element of Γ can be expressed in the form:

$$\gamma_1 p_1(\rho) + \gamma_2 p_2(\rho) + \dots + \gamma_n p_n(\rho)$$

(ρ operates on the right), where $\gamma_1, \gamma_2, \dots, \gamma_n$ are fixed elements of Γ and p_1, p_2, \dots, p_n are polynomials with integer coefficients.

Theorem A is proved now for an endomorphism ρ of an abelian group H whose dual group Γ is finitely generated with respect to ρ .

LEMMA 1 (Rochlin [2], § 3.4, see Appendix C.1).

If an abelian group Γ is finitely generated with respect to an endomorphism ρ and Ω is a ρ invariant subgroup, then Ω is finitely generated with respect to the restriction of ρ to Ω .

LEMMA 2 (Rochlin [2], § 4.2, see Appendix C.2).

If τ is an automorphism of an abelian group H whose dual group Γ is finitely generated with respect to τ , then H contains a sequence $H = H_0 \supset H_1 \supset H_2 \supset \dots$ of

closed subgroups such that $\bigcap_n H_n = e$, $\tau^{-1}H_n \subset H_n$ and H/H_n is the direct sum of a finite group and a finite dimensional torus for every n .

Theorem A for $(\tau^{-1})_{H/H_n}$ (as in Lemma 2) follows from Theorem A for a finite group (§ 3.2) and for a finite dimensional torus (§ 3.4) using proof by steps. Theorem A for τ^{-1} then follows by taking the limit (§ 4.1, (ii)). Theorem A for τ is then given by § 2.6, (iii) (as long as T is invertible).

Returning to proving Theorem A for the endomorphism ρ , ρ is replaced by an automorphism as in § 2.6, (i), T being replaced by the invertible transformation T_α ; Lemma 1 ensures that the factor group introduced by this process has a dual which is finitely generated with respect to τ . The result of the previous paragraph completes the proof.

LEMMA 3 (Rochlin [2], § 4.3, see Appendix C.3).

If ρ is an endomorphism of a compact separable abelian group H , then H contains a sequence

$H = H_0 \supset H_1 \supset H_2 \supset \dots$ of ρ invariant closed subgroups

such that $\bigcap_n H_n = e$ and the dual group of H/H_n is finitely generated with respect to ρ_{H/H_n} for every n .

Theorem A for ρ_{H/H_n} (as in Lemma 3) is given by the preceding work; Theorem A for any endomorphism ρ of an abelian group follows by taking the limit (§ 4.1, (ii)). This gives Theorem A for σ_Z and so completes the proof of Theorem A for a general endomorphism.

5. APPLICATIONS OF THEOREM A

5.1. GROUP ENDOMORPHISMS. If M is put equal to the group G and T to σ , then Theorem A states: a weakly mixing group endomorphism has completely positive entropy. An ergodic group endomorphism is known to be mixing ([4], p 54 and [10], § 6) and so it follows that an ergodic group endomorphism of a compact separable group has completely positive entropy. This is Yuzvinskii's result [1] mentioned in the introduction.

Theorem A can be applied to affine transformations in a similar sort of way to give: a weakly mixing affine group transformation has completely positive entropy.

5.2. SKEW-PRODUCTS. More generally, Theorem A can be applied to the following type of skew-product transformation: M is taken to be a direct product of an arbitrary Lebesgue space (X, \mathcal{C}, ν) and a compact separable group G with Borel sets and Haar measure m (the latter is also a Lebesgue space ; $\nu(X) = m(G) = 1$); T is given by: $T(x, y) = (Sx, \sigma(y)\varphi(x))$, where S is a measure preserving transformation of X and φ is a

measurable map from X to G ; T will be said to be a skew-product transformation with base S and fibres of type σ (σ being a group endomorphism of G).

G -action is given by: $g.(x,y) = (x,gy)$ and so

$Tg = \sigma gT$, i.e. $T\sigma$ -commutes with G -action. $T\xi_{(G)}$ is isomorphic to S and so if T is weakly mixing and S has completely positive entropy, then Theorem A implies that T has completely positive entropy.

5.5. NILMANIFOLDS. Suppose that T' is a group automorphism of a connected and simply connected nilpotent Lie group N (lower central series:

$N = N_0 \supset N_1 \supset N_2 \supset \dots \supset N_{k-1} \supset N_k = e$) which takes a uniform discrete subgroup D onto itself; the (left) coset space N/D is a compact manifold known as a nilmanifold. T' induces a measure preserving transformation T on N/D known as an 'automorphism'.

This type of transformation has been studied by W. Parry [7] who proved that an ergodic automorphism of a nilmanifold has completely positive entropy. This can be done as follows.

Let T_r be the transformation induced on $N/N_r D$ and let σ_r be the transformation induced on $N_{r-1} D/N_r D$.

$N_{r-1}D/N_rD$ is a torus and σ_r is a group automorphism of it. The ergodicity condition turns out to be:

T is ergodic if and only if T_1 ('the torus part') is ergodic. Assume now that T is ergodic; T is weakly mixing since the torus part of $T \otimes T$ is $T_1 \otimes T_1$, i.e. the cartesian square of T is ergodic ([4], p 39).

$T_1 = \sigma_1$ is just an ergodic automorphism of a torus and so has completely positive entropy. T_2 can be regarded as being a skew-product transformation with base T_1 and fibres of type σ_2 and so has completely positive entropy by § 5.2. T_3 can be regarded as a skew-product transformation with base T_2 and fibres of type σ_3 etc. So T_r has completely positive entropy for $r = 1, 2, \dots, k$; in particular, $T_k = T$ has completely positive entropy.

APPENDIX A

YUZVINSKII ([1], § 11.5). If ρ is an automorphism of a compact separable totally disconnected group H and A is an open normal subgroup such that $\bigcap_{-\infty}^{\infty} \rho^n A = e$ and H/A is simple, then either

(i) H is finite or

(ii) ρ is a Bernoulli group automorphism of H .

PROOF (Modified slightly). Let $F = H/A$ and

$\hat{H} = \bigotimes_{-\infty}^{\infty} F_i$, where each F_i is isomorphic to F . Let $\hat{\rho}$ be the Bernoulli group automorphism of \hat{H} and $p: H \rightarrow F$ be the natural projection with respect to A .

A map $Q: H \rightarrow \hat{H}$ is constructed:

$$Qh = \{p(\rho^{-i}h)\}_{-\infty}^{\infty};$$

clearly, $Q\rho = \hat{\rho}Q$. Q is clearly 1 - 1; if Q is onto, then ρ is isomorphic to $\hat{\rho}$ as group automorphisms. Suppose that $Q(H)$ is a proper subgroup of \hat{H} , i.e.

there exists a sequence $\{x_i\}_{-\infty}^{\infty}$ of elements in F such that $\bigcap_{-\infty}^{\infty} \rho^i(p^{-1}x_i)$ is empty. Compactness implies that a finite intersection is empty. Suppose that \bigcap_a^b is empty but that \bigcap_{a+1}^b is not. Then applying

ρ^{-a} and multiplying by x_a^{-1} , this becomes $A \cap D$ is empty, where D is a coset of the open normal subgroup $E = \bigcap_{i=1}^{b-a} \rho^i A$. Thus $AE \neq H$ and so the simplicity of H/A implies that $E \subset A$. Hence $\rho E = E$ and so $E \subset \bigcap_{n=-\infty}^{\infty} \rho^n A = e$. This implies that e is open and consequently that G is finite.

APPENDIX B

YUZVINSKII'S RESULTS ON COMPACT CONNECTED GROUPS

LEMMA ([1], § 3.3). If C is a compact connected separable group and Z is its centre, then the factor group C/Z is the direct product of algebraically simple Lie groups. In particular, C/Z has trivial centre.

PROOF. Andre Weil in 'L'Integration dans les Groups Topologiques et ses Applications' proved that C is of the form $(A \otimes B)/N$, where A is a connected abelian group isomorphic to a subgroup of Z , B is a direct product of simple Lie groups and N is a subgroup of the centre of $A \otimes B$. The centre of a direct product is

the direct product of the centres of the factors and so $Z = (A \otimes Z')/N$, where Z' is the centre of B . So C/Z is isomorphic to $(A \otimes B)/(A \otimes Z')$ which is isomorphic to B/Z' . A compact simple Lie group, when factored by its centre, is algebraically simple and so B/Z' is a direct product of such groups. Note that separability implies that the direct product is countable.

NOTE. It is well known in algebra that if a group is the direct product of two non-abelian simple groups, then the only non-trivial normal subgroups are the two factors.

THEOREM ([1], § 4.1). If τ is an automorphism of a compact separable group G whose centre is trivial, then $\tau = \tau_1 \otimes \tau_2$, where τ_1 is a Bernoulli group automorphism of G_1 and τ_2 is a direct product of automorphisms of τ -invariant semi-simple Lie groups; $G = G_1 \otimes G_2$.

PROOF. By the lemma, G is a countable direct product of algebraically simple Lie groups: $G = \bigotimes_n L_n$. Let p_i denote the projection of G onto L_i and p_{ij} the

projection onto $L_i \otimes L_j$. Consider τL_1 ; this must be a simple normal subgroup of G . $p_i(\tau L_1)$ can only be e or L_i . For some a , $p_a(\tau L_1)$ must equal L_a . Suppose also that $p_b(\tau L_1) = L_b$. Then $p_{ab}(\tau L_1)$ is a normal subgroup of $L_a \otimes L_b$ and so must equal $L_a \otimes L_b$ (by the note) contradicting the simplicity of τL_1 . So τ can only permute the L_i 's.

G has been shown to be the direct product of L_i cycles (under τ): G_1 is put equal to the direct product of all the infinite cycles and G_2 to the direct of all the finite cycles. The group of states for τ_1 is taken to be the direct product of a simple factor from each infinite cycle.

APPENDIX C

ROCHLIN'S RESULTS ON ABELAIN GROUPS

C.1 ([2], § 3.4). If an abelain group Γ is finitely generated with respect to an endomorphism ρ and Ω is a ρ -invariant subgroup, then Ω is finitely generated with respect to ρ restricted to Ω .

PROOF. Every element of Γ is of the form

$$\gamma_1 p_1(\rho) + \gamma_2 p_2(\rho) + \dots + \gamma_n p_n(\rho)$$

(see § 4.4). The result is a special case of a well known theorem in algebra: if the ring R is Noetherian, then any R -module with a finite number of generators is a Noetherian module. In the case given, R is the ring of polynomials with integer coefficients; R is Noetherian by a theorem of Hilbert.

Reference: Van der Waerden, 'Modern Algebra'.

C.2 ([2], § 4.2). If τ is an automorphism of an abelian group H whose dual group Γ is finitely generated with respect to τ , then H contains a sequence $H = H_0 \supset H_1 \supset H_2 \supset \dots$ of closed subgroups such that $\bigcap_n H_n = e$, $\tau^{-1} H_n \subset H_n$ and H/H_n is the direct sum of a finite group and a finite dimensional torus.

PROOF. Every element of Γ is of the form

$$\gamma_1 p_1(\tau) + \gamma_2 p_2(\tau) + \dots + \gamma_m p_m(\tau)$$

(see § 4.4). Let Ω_n be the subgroup of Γ obtained by limiting the degrees of the polynomials to n .

For all n greater than or equal to some N , Ω_n will contain $\gamma_1 \tau^{-1}, \gamma_2 \tau^{-1}, \dots, \gamma_m \tau^{-1}$ and so will be

invariant under τ^{-1} . The annihilator F_n of \mathcal{L}_n is invariant under τ^{-1} for $n \geq N$ and as \mathcal{L}_n is finitely generated, H/F_n is a direct product of a finite group and a finite dimensional torus. As $\bigcup_n \mathcal{L}_n = \Gamma$, $\bigcap_n F_n = e$. So H_1 is put equal to F_N , H_2 to F_{N+1} etc.

C.5 ([2], § 4.3). If ρ is an endomorphism of a compact separable abelian group H , then H contains a sequence $H = H_0 \supset H_1 \supset H_2 \supset \dots$ of ρ invariant closed subgroups such that $\bigcap_n H_n = e$ and the dual group of H/H_n is finitely generated with respect to ρ_{H/H_n} for every n .

PROOF. As H is separable, its dual group Γ is countable; so the elements of Γ can be indexed $\gamma_1, \gamma_2, \dots$. Let \mathcal{L}_n be the smallest subgroup of Γ containing $\gamma_1, \gamma_2, \dots, \gamma_n$ which is invariant under ρ and let H_n be the annihilator of \mathcal{L}_n . $\bigcup_n \mathcal{L}_n = \Gamma$ implying that $\bigcap_n H_n = e$ and so these H_n 's satisfy the given conditions.

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PAPER III

THE ADDITION THEOREM FOR THE ENTROPY OF
TRANSFORMATIONS OF G-SPACES

INTRODUCTION

In a previous paper [4], conditions were given for a certain type of transformation of a G -space (G being a compact separable group) to have completely positive entropy. It is useful to be able to calculate the actual numerical value of the entropy; the purpose of the present paper is to extend previously known formulae to cover the type of transformation just mentioned.

As in [4], the notation of Rochlin's survey article [3] is used: the entropy of a measure preserving transformation T of a Lebesgue space (M, \mathcal{B}, μ) is denoted by $h(T)$; $H(\xi)$ denotes the entropy of the (measurable) partition ξ of M and $H(\xi/\gamma)$ denotes the mean conditional entropy of ξ with respect to γ .

Throughout this paper, the basic measure space (M, \mathcal{B}, μ) will be a direct product of a Lebesgue space (X, \mathcal{E}, ν) and a compact separable group G with Borel sets and Haar measure m (this also being a Lebesgue space); all the measures are normalized

i.e. $\mu(M) = \nu(X) = m(G) = 1$. The measure preserving transformation T will act as follows:

$$T(x, g) = (Sx, \sigma(g)\phi(x)) ,$$

where S is a measure preserving transformation of X , σ is a group endomorphism of G and $\phi : X \rightarrow G$ is some measurable map; throughout this paper, such a transformation will be described as a skew-product of S and σ (the map ϕ not being specified).

It will be proved that

$$h(T) = h(S) + h(\sigma) . \quad \dots (1)$$

For the case where M is itself a compact separable group, T is a group endomorphism and G is a T invariant ($TG \subset G$) closed normal subgroup ($\sigma =$ the restriction of T to G), this result was proved by Yuzvinskii in [2] as an essential step in his proof that an ergodic endomorphism of a compact separable group has completely positive entropy - the result generalized by the present author in [4]; (1) was not needed there but it is a very useful result in its own right.

The generalization of Yuzvinskii's result has essentially been made possible by Section 2.5 (a compact connected Lie group is 'rigid'). This also leads to a simplification of the proof as does a different treatment of Bernoulli endomorphisms (§ 3). Apart from this, Yuzvinskii's work is adapted to fit the broader context.

Note that more restrictive conditions are given here for T than were given in [4] (skew-product as opposed to σ -commuting with G -action); this is necessary for the elimination of trivial group action (if this were allowed, σ and consequently $h(\sigma)$ could be practically anything for a given system) and does not seriously limit applications.

Following Yuzvinskii, (1) will be known as 'The Addition Theorem'.

1. PRELIMINARIES

1.1. STANDARD RESULTS FROM ENTROPY THEORY.

(i) If $\xi_1 \leq \xi_2 \leq \dots$ is a sequence of T invariant $(T^{-1}\xi_n \leq \xi_n)$ partitions of M such that $\bigvee_n \xi_n = \xi$, then $h(T_{\xi_1}) \leq h(T_{\xi_2}) \leq \dots$ and $\lim_{n \rightarrow \infty} h(T_{\xi_n}) = h(T)$, where T_{ξ_n} is the factor-transformation induced by T in the Lebesgue space M/ξ_n consisting of elements of ξ_n ([3], § 9.6). (ξ always denotes the partition of the space being considered into distinct points.)

(ii) For any two partitions ξ and η with finite entropy (in particular, if the partitions are finite),

$$|H(\xi) - H(\eta)| \leq H(\xi/\eta) + H(\eta/\xi) \quad ([3], \S 6.5).$$

(iii) For any three partitions ξ , η and ζ ,

$$h(\xi \vee \eta / \zeta) \leq H(\xi / \zeta) + H(\eta / \zeta) \quad ([3], \S 5.6),$$

$$H(\xi / \eta \vee \zeta) \leq H(\xi / \zeta) \quad ([3], \S 5.10).$$

$$(iv) \quad h(T^n) = nh(T) \quad ([3], \S 9.3).$$

(v) In [1], Abramov and Rochlin proved that the entropy of any skew-product transformation is given by:

$$h(T) = h(S) + h_S(\tau), \quad \dots (2)$$

where T and S are as above, $\tau = \{\tau_x\}$ is a collection of measure preserving transformations of the second factor - for the purposes of this paper,

$$\tau_x(g) = \sigma(g)\varphi(x)$$

and $h_S(\tau)$ is a quantity known as the 'mixed entropy' of τ .

If $\xi_1 \leq \xi_2 \leq \dots$ is an increasing sequence of finite partitions of the second factor (G in this case)

such that $\bigvee_n \xi_n = \xi$, then

$$h_S(\tau) = \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} \int_X H(\xi_{x,m}^k) d\nu, \quad \dots (3)$$

$$\text{where } \xi_{x,m}^k = \bigvee_{i=0}^{k-1} \tau_x^{-i} \cdot \tau_{Sx}^{-i} \cdot \dots \cdot \tau_{S^{i-1}x}^{-i} \xi_m.$$

(3) should be compared with the following formula for $h(\sigma)$ to be used later:

$$h(\sigma) = \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} H(\xi_m^k), \quad \dots (4)$$

where $\xi_m^k = \bigvee_{i=0}^{k-1} \sigma^{-i} \xi_m$ and the sequence $\{\xi_n\}$ is as before.

(4) follows from [3], § 9.5 and § 7.3.

Clearly, formula (3) will be very important in

the proof of the addition theorem - all that remains is to prove that $h_S(\tau) = h(\sigma)$. Alas, this is not as easy as it might at first appear. If T is a direct product (i.e. $\phi(x) = e$ for all x in X), then $h_S(\sigma) = h(\sigma)$ ([1], § 2.4).

1.2. SUBGROUPS OF G . Let H be a completely σ -invariant ($\sigma H = H$) closed subgroup of G . $\mathcal{S}(H)$ denotes the partition of M each element of which is the direct product of a point from X with a right coset of H ; σ_H denotes the restriction of σ to H ; $\sigma_{G/H}$ denotes the transformation induced by σ on the right coset space G/H and $T_{\mathcal{S}(H)}$ denotes the factor-transformation induced by T on $M/\mathcal{S}(H)$, $\mathcal{S}(H)$ being invariant under T .

There exists a Borel cross-section ψ which takes almost every coset Hg to a single point $\psi(g) \in Hg$; $\psi(G)$ is a Borel subset $Y \subseteq G$. Measure theoretically, G is the direct product of Y and H (Y having the measure induced by m and H having its own normalized Haar measure). $\psi \cdot \sigma$ is isomorphic to and will be identified with $\sigma_{G/H}$. Thus σ becomes the skew-product of $\sigma_{G/H}$ and σ_H . Similarly, as M is the direct

product of $(X \otimes Y)$ and H , T can also be regarded as being a skew-product of $T_{\xi(H)}$ and σ_H .

In order to deal with situations in which σ is not onto (when a subgroup is properly mapped into itself), Yuzvinskii in [2] defined $h(\sigma)$ by putting $h(\sigma) = h(\sigma_{G'})$, where $G' = \bigcap_n \sigma^n G$. It turns out that $h(S) = h(T_{\xi(G')})$ and so one could say that $h(T) = h(S) + h(\sigma)$ if one knew that $h(T) = h(T_{\xi(G')}) + h(\sigma_{G'})$. All this goes over to generalization but it is felt that its value is limited - it does not make much sense to consider the entropy of a non-measure-preserving transformation. As it is sufficient to consider only the entropy of endomorphisms restricted to completely invariant subgroups, only skew-products of a measure preserving transformation and a group endomorphism mapping a group onto itself will be considered.

1.3. PROOF BY STEPS. Let H be a closed completely invariant normal subgroup of G . Clearly, $T_{\xi(H)}$ is a skew-product of S and $\sigma_{G/H}$.

Suppose that the addition theorem has been proved for endomorphisms of the types of $\sigma_{G/H}$ and σ_H or, as

will be stated in future; that one has the addition theorem for $\sigma_{G/H}$ and for σ_H (no limitations on S), then

$$h(T_{\xi(H)}) = h(S) + h(\sigma_{G/H}) ,$$

$$h(T) = h(T_{\xi(H)}) + h(\sigma_H) \quad \text{and}$$

$$h(\sigma) = h(\sigma_{G/H}) + h(\sigma_H) .$$

Hence,

$$h(T) = h(S) + h(\sigma_{G/H}) + h(\sigma_H) = h(S) + h(\sigma) .$$

So, the addition theorem for σ can be proved by finding a suitable completely σ invariant closed normal subgroup H and then proving it for the two 'steps' $\sigma_{G/H}$ and σ_H .

This principle can clearly be extended to a finite number of steps: suppose that G contains a sequence $G = G_0 \supset G_1 \supset G_2 \supset \dots \supset G_n = e$ of completely σ invariant closed subgroups with G_{i+1} normal in G_i ; the addition theorem for σ can be proved by proving it for all the steps $\sigma_{G_i/G_{i+1}}$. The next subsection shows that the number of steps can be infinite.

1.4. TAKING LIMITS. Suppose that G contains a sequence $G = G_0 \supset G_1 \supset G_2 \supset \dots$ of σ invariant ($\sigma G_n \subset G_n$) closed subgroups such $\bigcap_n G_n = e$ and that

$$h(T_{\xi(G_n)}) = h(S) + h(\sigma_{G/G_n}) \text{ for all } n. \quad \dots (5)$$

By 1.1, (i),

$$h(T) = \lim_{n \rightarrow \infty} h(T_{\xi(G_n)}) \quad \text{and}$$

$$h(\sigma) = \lim_{n \rightarrow \infty} h(\sigma_{G/G_n}).$$

Hence,

$$h(T) = h(S) + h(\sigma).$$

So (5) implies the addition theorem for σ .

2. RIGID GROUPS

2.1. NOTATION. Suppose that $\xi = \{A_\alpha\}$ is a partition of a measurable set $B \subset G$.

For any g in G , ξg will denote the partition of Bg into $\{A_\alpha g\}$.

For a measurable set $C \subset G$, $\xi - C$ will denote the partition of $B - C$ into $\{(A_\alpha - C)\}$.

2.2. DEFINITION. The group G is said to be rigid if there exist an increasing sequence $\xi_1 \leq \xi_2 \leq \dots$ of finite partitions of G and a real number Q such that $\bigvee_n \xi_n = \xi$ and $H(\xi_n g / \xi_n) \leq Q$ for all n .

2.3. THEOREM. The addition theorem holds for an endomorphism σ of a rigid group G .

PROOF (adapted from Yuzvinskii [2], § 7.4).

By § 1.1, (v), it is sufficient to prove that

$h_S(\tau) = h(\sigma)$. As

$$T(x, g) = (Sx, \sigma(g)\varphi(x)) = (Sx, \tau_x(g)) \quad (\tau = \{\tau_x\}),$$

T^n can be expressed in the form:

$$T^n(x, g) = (S^n x, \sigma^n(g)\varphi_n(x)) = (S^n x, \tau_{n,x}(g))$$

$$(\tau_n = \{\tau_{n,x}\}).$$

Now $h(T^n) = h(S^n) + h_{S^n}(\tau_n)$ (applying formula for the entropy of a skew-product, § 1.1, (v)) and so $nh(T) = nh(S) + h_{S^n}(\tau_n)$ (applying § 1.1, (iv)).

Hence,

$$h_{S^n}(\tau_n) = nh_S(\tau) \quad \dots (6)$$

(comparing the previous formula with (2)).

$$\tau_{n,x}(g) = \sigma^n(g)\varphi_n(x)$$

and so

$$\tau_{n,x}^{-1}\gamma = \sigma^{-n}(\gamma.[\varphi_n(x)]^{-1}) \text{ for any partition } \gamma \text{ of } G.$$

Applying formulae (3) and (4) of 1.1, (v),

$$\begin{aligned} |h_{S^n}(\tau_n) - h(\sigma^n)| &= \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} \left| \int_X (H(\xi_{x,m}^{k,n}) - H(\xi_m^{k,n})) d\nu \right| \\ &\leq \lim_{m \rightarrow \infty} \lim_{k \rightarrow \infty} \frac{1}{k} \int_X \Delta_{x,m}^{k,n} d\nu, \end{aligned}$$

$$\text{where } \Delta_{x,m}^{k,n} = H(\xi_{x,m}^{k,n}) - H(\xi_m^{k,n}),$$

$$\xi_{x,m}^{k,n} = \bigvee_{i=0}^{k-1} \tau_{n,x}^{-1} \cdot \tau_{n,Sx}^{-1} \cdot \dots \cdot \tau_{n,S^{i-1}x}^{-1} \xi_m$$

($\{\xi_m\}$ as in Definition 2.2)

$$\begin{aligned} &= \bigvee_{i=0}^{k-1} \sigma^{-ni}(\xi_m.[\varphi_n(S^{i-1}x)]^{-1} \cdot \dots \cdot [\varphi_n(x)]^{-1}) \\ &= \bigvee_{i=0}^{k-1} \sigma^{-ni}(\xi_m.f_{n,i}(x)) \text{ (simplifying notation),} \end{aligned}$$

$$\xi_m^{k,n} = \bigvee_{i=0}^{k-1} \sigma^{-ni} \xi_m.$$

Using 1.1, (ii),

$$\begin{aligned} \Delta_{x,m}^{k,n} \leq & H\left(\bigvee_{i=0}^{k-1} \sigma^{-ni} (\xi_m \cdot f_{n,i}(x)) / \bigvee_{i=0}^{k-1} \sigma^{-ni} \xi_m\right) + \\ & + H\left(\bigvee_{i=0}^{k-1} \sigma^{-ni} \xi_m / \bigvee_{i=0}^{k-1} \sigma^{-ni} (\xi_m \cdot f_{n,i}(x))\right), \end{aligned}$$

which can be expanded by 1.1, (iii) to give:

$$\begin{aligned} \Delta_{x,m}^{k,n} \leq & \sum_{i=0}^{k-1} \left[H(\sigma^{-ni} (\xi_m \cdot f_{n,i}(x)) / \sigma^{-ni} \xi_m) + \right. \\ & \left. + H(\sigma^{-ni} \xi_m / \sigma^{-ni} (\xi_m \cdot f_{n,i}(x))) \right] \\ = & \sum_{i=0}^{k-1} \left[H(\xi_m \cdot f_{n,i}(x) / \xi_m) + H(\xi_m \cdot [f_{n,i}(x)]^{-1} / \xi_m) \right] \end{aligned}$$

$\leq 2kQ$ by the definition of a rigid group.

Hence, it follows that

$$|h_{S^n}(\tau_n) - h(\sigma^n)| \leq 2Q.$$

This combined with (6) and § 1.1, (iv) gives:

$$|h_S(\tau) - h(\sigma)| \leq 2Q/n \quad \text{for all } n.$$

So. $h_S(\tau)$ must equal $H(\sigma)$.

2.4. TOTALLY DISCONNECTED GROUPS. A search for rigid groups is begun now. It is easy to prove that a compact separable totally disconnected group G is rigid:

PROOF (Yuzvinskii [2], § 7.2). G , being totally disconnected, contains a sequence $G = G_0 \supset G_1 \supset G_2 \supset \dots$ of open normal subgroups such that $\bigcap_n G_n = e$. Let ξ_n be the partition of G into cosets of G_n ; then $\xi_n g = \xi_n$ for all g in G and all n . Therefore, $H(\xi_n g / \xi_n) = 0$ for all g and all n and so $\xi_1 \leq \xi_2 \leq \dots$ satisfies all the requirements of Definition 2.2 with $Q = 0$.

2.5. LIE GROUPS. A compact Lie group also turns out to be rigid but this is more difficult to prove. It is however a crucial result. First some simple observations.

NOTE. The maximum number of p -dimensional (open) balls of radius r that can intersect a single p -ball of radius R depends only on p and the ratio $r/R = s$. This number will be denoted by $I(p, s)$.

DEFINITION. A sequence $\xi_1 \leq \xi_2 \leq \dots$ of finite partitions of the group G such that $\bigvee_n \xi_n = \mathcal{E}$ will be said to satisfy the 'bounded intersection condition' if the number of elements of $\xi_n g$ which intersect a single element of ξ_m (in sets of positive measure) is bounded above by some number N for all g in G and all n .

LEMMA. The existence of a sequence of partitions $\{\xi_n\}$ satisfying the bounded intersection condition implies that G is rigid.

PROOF. For a finite partition $\xi = \{A_i\}$ of G and g in G , the definition of conditional entropy gives:

$$H(\xi g / \xi) = \sum_i m(A_i) \cdot H(\xi g, A_i),$$

$$\text{where } H(\xi g, A_i) = - \sum_j \frac{m(A_j g \cap A_i)}{m(A_i)} \log \frac{m(A_j g \cap A_i)}{m(A_i)}$$

is the entropy of the partition of A_i into the collection of sets $\{A_j g \cap A_i\}_j$.

$H(\xi g, A_i) \leq \log m_i$, where m_i is the number of elements of the partition $\{A_j g \cap A_i\}_j$ (sets of measure zero not being counted) ([3], § 4.7).

It follows that $H(\xi_g/\xi) \leq \log m$, where $m = \max_i m_i$.

Hence for the sequence $\{\xi_n\}$ satisfying the conditions of the definition, $H(\xi_n g/\xi_n) \leq \log N$ for all n and all g in G and so the conditions of Definition 2.2 are satisfied with $Q = \log N$.

THEOREM. Any compact Lie group G admits a sequence $\xi_1 \leq \xi_2 \leq \dots$ of finite partitions such that $\bigvee_n \xi_n = \xi$ and the bounded intersection condition is satisfied.

PROOF. First it is assumed that G is connected.

G has a Riemannian structure which is invariant under both left and right translation ([5], p 188) and the resulting metric d has the following property ([5], Ch. I, Propositions 9.9 and 9.10):

There exists an $\varepsilon > 0$ such that, for A and B in the tangent plane TG_e at the identity and $\|A\| < \varepsilon$ and $\|B\| < \varepsilon$, N_ε is a normal neighbourhood of e and

$$\frac{\|A - B\|}{d(a, b)} \longrightarrow 1 \quad \text{as} \quad (a, b) \longrightarrow (e, e),$$

where $a = \text{Exp}_e A$, $b = \text{Exp}_e B$ and N_ε is the spherical neighbourhood of e of radius ε (w.r.t. d).

δ is chosen now sufficiently small (smaller than $\varepsilon/4\sqrt{p}$, where p is the dimension of G) for

$$\frac{1}{2} < \frac{A - B}{d(a, b)} < 2 \quad \text{for } a \text{ and } b \text{ in } N_{4\delta\sqrt{p}}.$$

A (open) cube C of edge δ with one corner at O ($\text{Exp}_e O = e$) is constructed in TG_e . C_n will denote the partition of C into $(n!)^p$ equal (open) cubes C_{n1}, C_{n2}, \dots each of edge $\delta/n!$ (a set of measure zero has been discarded). Let $D = \text{Exp}_e C$ and $D_n = \text{Exp}_e C_n$ (i.e. D_n is the partition of D into $\{\text{Exp}_e C_{ni}\}$).

As G is compact, a finite number of translations of D ($D, Da_1, Da_2, \dots, Da_{m-1}$ say) cover G .

The partition ξ_n is formed by taking the elements of $D_n, D_n a_1 - D, D_n a_2 - (D \cup Da_1), \dots,$

$$D_n a_{m-1} - (D \cup Da_1 \cup \dots \cup Da_{m-2});$$

in other words, m copies of D_n are fitted together so that there is no overlapping (notation as in § 2.1).

Clearly, $\{\xi_n\}$ is an increasing sequence of finite partitions satisfying $\bigvee_n \xi_n = \xi$; it remains to show that $\{\xi_n\}$ satisfies the bounded intersection condition.

The intersection of $D_n g$ with D_n for any g in G is considered: if this intersection is non-empty, $D_n g \subset N_{4\delta\sqrt{p}}$ and then $E_n(g) = \text{Exp}_e^{-1} D_n g$ can be considered:

Each element of C_n contains a ball of diameter $\delta/n!$ and is contained in a ball of diameter $\delta\sqrt{p}/n!$.

Hence, each element of D_n contains a ball of diameter $\delta/n!^2$ and is contained in a ball of diameter $2\delta\sqrt{p}/n!$.

As d is invariant under translation, the elements of $D_n g$ have the same property and so if $E_n(g)$ is defined, then each element of $E_n(g)$ contains a ball of diameter $\delta/n!^4$ and is contained in a ball of $4\delta\sqrt{p}/n!$

It follows that not more than

$$I(p, \frac{\delta}{n!^4} / \frac{\delta\sqrt{p}}{n!}) = I(p, 1/4\sqrt{p})$$

elements of $E_n(g)$ can intersect a single element of C_n and hence that not more than $I(p, 1/4\sqrt{p})$ elements of $D_n g$ can intersect a single element of D_n .

As ξ_n is constructed from m copies of D_n , not more than $mI(p, 1/4\sqrt{p})$ elements of $\xi_n g$ can intersect a single element of ξ_n for all g in G and all n .

Thus $\{\xi_n\}$ satisfies the bounded intersection condition.

The extension of the proof to a finite number of connected components is trivial.

2.6. ABELIAN GROUPS OF FINITE DIMENSION. Finite-dimensional tori were covered by the last section; the following theorem also includes a wider class of abelian groups.

THEOREM (Yuzvinskii [2], § 7.3). A compact connected abelian group G of finite dimension p is rigid.

PROOF (adapted from [2]). G contains a sequence

$G = G_0 \supset G_1 \supset G_2 \supset \dots$ of closed subgroups such that

$\bigcap_m G_m = e$ and G/G_m is isomorphic to some p -dimensional

torus A for all m . Let $P_m: G \rightarrow A$ be the map obtained by composing the projection of G onto G/G_m with the isomorphism of G/G_m onto A .

Using the notation of the proof of the last theorem, Exp_e is an isometry for a torus and D is taken to be the usual representation of A as a p -cube. The partitions $\{P_m^{-1}D_n\}$ are finite and satisfy:

$$H((P_m^{-1}D_n)g/P_m^{-1}D_n) = H(D_n \cdot P_m g/D_n) \leq \log I(p, 1/4\sqrt{p}).$$

(see proof of lemma in § 2.5).

An increasing sequence of integers $n(1) = 1$, $n(2) = 2$, $n(3)$, $n(4)$, ... is constructed inductively so that $\{\xi_m = P^{-1}_{n(m)} D_{n(m)}\}$ is an increasing sequence; it is assumed that all the $n(r)$'s have been chosen for $r \leq k$. $P_k P_{k+1}^{-1}$ is an endomorphism of A onto A and so has a kernel of finite order s ; the inverse map $P_{k+1} P_k^{-1}$ takes a single cube from $D_{n(k)}$ to the union of s disjoint identical cuboids. It is easy to see that $D_{sn(k)} \geq P_{k+1} P_k^{-1} D_{n(k)}$ from which it follows that $P_{k+1}^{-1} D_{sn(k)} \geq P_k^{-1} D_{n(k)}$. So $n(k+1)$ is put equal to $sn(k)$.

The sequence $\{\xi_m\}$ so constructed satisfies all the conditions of Definition 2.2 and so G is rigid.

3. BERNOULLI GROUP AUTOMORPHISMS AND ENDOMORPHISMS

In the last section, the addition theorem was proved for endomorphisms of totally disconnected groups, Lie groups and finite-dimensional abelian groups. In order to be able to use the theory of the structure of compact groups to tie these results together to give the addition theorem for an endomorphism of an arbitrary compact group G , it is necessary to prove the addition theorem for one more class of endomorphisms, namely Bernoulli group automorphisms and endomorphisms.

3.1. DEFINITIONS. Let G be a direct product of a two-way (one-way) infinite sequence of copies of some compact group G_0 known as the group of states; an element g can be represented as a sequence $\{g_i\}_{-\infty}^{\infty}$ ($\{g_i\}_0^{\infty}$). A Bernoulli group automorphism (endomorphism) takes the sequence $\{g_i\}_{-\infty}^{\infty}$ ($\{g_i\}_0^{\infty}$) to the sequence $\{h_i\}_{-\infty}^{\infty}$ ($\{h_i\}_0^{\infty}$), where $h_i = g_{i+1}$.

In some cases G will be rigid (if G_0 is finite for example) but the following theorem covers all possibilities.

3.2. THEOREM. When σ is a Bernoulli group endomorphism of the group G , T is isomorphic to the direct product of S and σ .

PROOF. $T(x, g) = (Sx, \sigma(g)\varphi(x))$; in this case, $\varphi(x)$ is a sequence $\{\varphi_i(x)\}_0^\infty$. Let $F: M \rightarrow M$ be the invertible measure preserving transformation given by: $F(x, g) = (x, g.f(x))$, where $f = \{f_i\}_0^\infty$ is some measurable function from X to G .

$$F^{-1}TF(x, g) = (Sx, \sigma(g)\sigma(f(x))\varphi(x)[f(Sx)]^{-1})$$

and so if

$$\sigma(f(x))\varphi(x)[f(Sx)]^{-1} = e \quad \dots (7)$$

for all x in X , then $S \otimes \sigma = F^{-1}TF$, i.e. T is isomorphic to $S \otimes \sigma$.

(7) is equivalent to the set of equations:

$$f_{i+1}(x)\varphi_i(x) = f_i(Sx). \quad \dots (8)$$

The equations (8) are satisfied by:

$$\begin{aligned} f_0(x) &= e ; \\ f_1(x) &= [\varphi_0(x)]^{-1} ; \\ f_2(x) &= [\varphi_0(Sx)\varphi_1(x)]^{-1} ; \\ f_i(x) &= [\varphi_0(S^{i-1}x)\varphi_1(S^{i-2}x)\dots\varphi_{i-1}(x)]^{-1}. \end{aligned}$$

As the f_i 's are all products of measurable functions, the measurability requirements for f and F are satisfied and so the map F , given by these solutions of (8), gives an isomorphism between T and $S \otimes \sigma$.

COROLLARY. For σ a Bernoulli group automorphism or endomorphism of the group G ,

$$h(T) = h(S) + h(\sigma).$$

PROOF. When σ is a Bernoulli group endomorphism, T is isomorphic to a direct product and so has the same entropy ([3], § 16.3):

$$h(T) = h(S \otimes \sigma) = h(S) + h(\sigma) \quad (\S 1.1, (v)).$$

For a Bernoulli group automorphism, let G_n be the closed normal subgroup of G consisting of all sequences $\{g_i\}_{i=-\infty}^{\infty}$ for which $g_k = e$ for $k \geq -n$, $n = 0, 1, \dots$. $\bigcap_n G_n = e$ and σ_{G/G_n} is a Bernoulli group endomorphism. So

$$h(T_{\{G_n\}}) = h(S) + h(\sigma_{G/G_n}) \text{ for all } n$$

and taking limits (§ 1.4) gives:

$$h(T) = h(S) + h(\sigma) .$$

4. COMPLETION OF PROOF

To complete the proof of the addition theorem, it is sufficient to show that an arbitrary compact group G breaks down into a sequence of factors groups on which endomorphisms of the types dealt with in the last two sections are induced; the 'proof by steps' procedure of § 1.3 can then be applied. Throughout this section, C will denote the connected component of the identity of G and Z will denote the centre of C ; both C and Z are completely invariant under any endomorphism σ of G .

4.1. TOTALLY DISCONNECTED GROUPS. The first step is easy: G/C is totally disconnected (addition theorem for $\sigma_{G/C}$ given by § 2.3 and § 2.4).

4.2. CONNECTED GROUPS WITH TRIVIAL CENTRES.

The next step will be $\sigma_{C/Z}$; this will require two results of Yuzvinskii [2] which are reproduced in [4], Appendix B. The first is that C/Z has a trivial centre and the second is the following:

LEMMA. An endomorphism ρ of a compact separable connected group H whose centre is trivial is the

direct product of a Bernoulli group automorphism ρ_1 , a Bernoulli group endomorphism ρ_2 and ρ_3 which is a direct product of automorphisms of semi-simple Lie groups.

The addition theorem has been proved for Bernoulli group automorphisms and endomorphisms (§ 3.2) and for endomorphisms of Lie groups (§ 2.3 and § 2.5). So the (infinite if necessary) proof by steps procedure gives the addition theorem for ρ as in the lemma. So the $\sigma_{C/Z}$ step is permissible. This leaves Z .

4.3. ABELIAN GROUPS. The endomorphism induced on Z factored by its connected component of the identity can be dealt with as in § 4.1 and so it remains to prove the addition theorem for an endomorphism σ of a connected abelian group G . The character group of G will be denoted by Γ .

First of all it is assumed that Γ is finitely generated with respect to σ , i.e. every element of Γ is of the form:

$$\gamma_1 p_1(\sigma) + \gamma_2 p_2(\sigma) + \dots + \gamma_n p_n(\sigma),$$

where $\gamma_1, \gamma_2, \dots, \gamma_n$ are fixed elements of Γ and

p_1, p_2, \dots, p_n are polynomials with integer coefficients. (Note that the adjoint of σ is denoted by the same symbol but written on the right.)

There are two possible cases:

(i) For every γ_i , there exists a polynomial q_i such that $\gamma_i q_i(\sigma) = 0$. In this case, Γ is of finite rank. So G is finite-dimensional and the addition theorem for σ is given by § 2.6.

(ii) For some γ_i , $\gamma_i q(\sigma) \neq 0$ for all possible polynomials q . Let Γ_i be the subgroup generated by γ_i , let Ω be the smallest subgroup containing Γ_i which is invariant under σ and let H be the annihilator of Ω . The condition given implies that

$\Gamma_i \sigma^a \cap \Gamma_i \sigma^b = e$ for $a \neq b$ and so Ω is the direct sum (finite numbers of non-zero terms) of

$\Gamma_i, \Gamma_i \sigma, \Gamma_i \sigma^2, \dots$, each of which is isomorphic to the integers. Its dual G/H is the direct sum

(unrestricted) of a one-way infinite sequence of circles and $\sigma_{G/H}$ is a Bernoulli group endomorphism with the circle as group of states.

$h(\sigma_{G/H}) = \infty$ ([3], § 9.10) and so the addition theorem for $\sigma_{G/H}$ (§ 3.2) gives:

$$h(T_{\mathcal{G}(H)}) = h(S) + h(\sigma_{G/H}) = \infty.$$

By 1.1, (i),

$$h(T) \geq h(T_{\mathcal{G}(H)}) = \infty;$$

$$h(\sigma) \geq h(\sigma_{G/H}) = \infty.$$

Hence,

$$h(T) = h(S) + h(\sigma).$$

This proof is a much modified version of [2], § 8.4.

The restriction on Γ is removed now.

LEMMA (Rochlin [6], § 4.3, reproduced in [4],

Appendix C.3). If σ is an endomorphism of a compact separable abelian group G , then G contains a sequence $G = G_0 \supset G_1 \supset G_2 \supset \dots$ of σ invariant closed subgroups such that $\bigcap_n G_n = e$ and the dual group of G/G_n is finitely generated with respect to σ_{G/G_n} for all n .

This lemma and the preceding work give

$$h(T_{\mathcal{G}(G_n)}) = h(S) + h(\sigma_{G/G_n}) \quad \text{for all } n$$

and on taking limits (§ 1.4), this becomes:

$$h(T) = h(S) + h(\sigma).$$

This completes the proof of the addition theorem.

5. APPLICATIONS

5.1. GROUP ENDOMORPHISMS. If σ is an endomorphism of a compact group G onto itself and H is a completely invariant closed subgroup, then σ can be written as a suitable skew-product (see § 1.2) and so

$$h(\sigma) = h(\sigma_{G/H}) + h(\sigma_H) .$$

For H normal, this is the result of Yuzvinskii [2] , mentioned in the introduction. The addition theorem can be applied in a similar fashion to affine transformations.

5.2. NILMANIFOLDS. The addition theorem is used by W. Parry [7] to calculate the entropy of an automorphism of a nilmanifold. Suppose that T' is a group automorphism of a connected and simply connected nilpotent Lie group N (lower central series $N = N_0 \supset N_1 \supset N_2 \supset \dots \supset N_{k-1} \supset N_k = e$) which takes a uniform discrete subgroup D onto itself; the (left) coset space N/D is a compact manifold known as a nilmanifold. T' induces a measure preserving transformation on N/D known as an 'automorphism'.

Let T_r be the transformation induced on $N/N_r D$ and let σ_r be the transformation induced on $N_{r-1} D / N_r D$. T_r can be regarded as a skew-product of T_{r-1} and σ_r for $r = 1, 2, \dots, k$ and so applying the addition theorem for each r , it follows that

$$h(T) = h(\sigma_1) + h(\sigma_2) + \dots + h(\sigma_r) .$$

σ_r is an automorphism of a torus and so

$$h(\sigma_r) = \sum_{|\lambda_i| > 1} \log |\lambda_i|$$

(see [4] for references), the λ_i 's being the eigenvalues of σ_r . Hence the entropy of T can be expressed in the form:

$$h(T) = \sum_{|\mu_i| > 1} \log |\mu_i| ,$$

where the μ_i 's are the eigenvalues of the differential of T at the identity coset of N/D .

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